

Linear wave systems on n -D spatial domains

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Abstract

In this paper we study the linear wave equation on an n -dimensional spatial domain. We show that there is a boundary triplet associated to the undamped wave equation. This enables us to characterise all boundary conditions for which the undamped wave equation possesses a unique solution non-increasing in the energy. Furthermore, we add boundary inputs and outputs to the system, thus turning it into an impedance conservative boundary control system.

Keywords: Wave equation, boundary triplet, boundary control

1 Introduction

In this paper we study the following linear system associated to the wave equation:

$$\left\{ \begin{array}{l} \rho(\xi) \frac{\partial^2 z}{\partial t^2}(\xi, t) = \operatorname{div} (T(\xi) \operatorname{grad} z(\xi, t)) - \left(Q_i \frac{\partial z}{\partial t} \right) (\xi, t), \quad \xi \in \Omega, \quad t \geq 0, \\ 0 = \frac{\partial z}{\partial t}(\xi, t) \quad \text{on } \Gamma_0 \times \mathbb{R}_+, \\ 0 = \nu \cdot (T(\xi) \operatorname{grad} z(\xi, t)) + \left(Q_b \frac{\partial z}{\partial t} \right) (\xi, t) \quad \text{on } \Gamma_1 \times \mathbb{R}_+, \\ u(\xi, t) = \nu \cdot (T(\xi) \operatorname{grad} z(\xi, t)) \quad \text{on } \Gamma_2 \times \mathbb{R}_+, \\ y(\xi, t) = \frac{\partial z}{\partial t}(\xi, t) \quad \text{on } \Gamma_2 \times \mathbb{R}_+, \\ z(\xi, 0) = z_0(\xi), \quad \frac{\partial z}{\partial t}(\xi, 0) = w_0(\xi) \quad \text{on } \Omega; \end{array} \right. \quad (1.1)$$

here $\Omega \subset \mathbb{R}^n$ is a bounded spatial domain with Lipschitz-continuous boundary $\partial\Omega = \overline{\Gamma_0} \cup \overline{\Gamma_1} \cup \overline{\Gamma_2}$, with $\Gamma_k \cap \Gamma_\ell = \emptyset$ for $k \neq \ell$. The vector ν denotes the outward normal at the boundary. Furthermore, $z(\xi, t)$ is the deflection

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from the equilibrium position at point $\xi \in \Omega$ and time $t \geq 0$, u (the forces on Γ_2) is the *input*, and y (the velocities at Γ_2) is the *output*. The physical parameters, $\rho(\cdot)$ and $T(\cdot)$ denote the mass density and Young's elasticity modulus, respectively. The operators Q_i and Q_b correspond to damping inside the domain Ω and at a part of its boundary, respectively. Typically Q_i and Q_b are point-wise multiplication operators, but they need not be.

Note that we do not assume that the sets Γ_k are separated, i.e., that $\overline{\Gamma_k} \cap \overline{\Gamma_\ell} = \emptyset$, $k \neq \ell$. However, we assume that the Γ_k 's are disjoint open subsets in the relative topology of the boundary, and that the boundaries $\partial\Gamma_k$ of the Γ_k 's have surface measure zero.

The wave system is a standard system in control of partial differential equations which has been widely studied before in the literature; see for instance [Paz83, Section 7.3], [RR93, Section 11.3.2], or [Yos95, Section XIV.3] for the zero-input case $u = 0$. Among the more recent papers which are closer to our treatment are [ALM13, MS06, MS07]. Compared to these, we allow a more general spatial domain, a more general boundary damping operator Q_b , and spatially varying physical parameters ρ and T .

A main difference between our treatment of the wave equation and those cited above is the first-order representation used in this study. We consider the semigroup generator $\begin{bmatrix} 0 & \text{div} \\ \text{grad} & 0 \end{bmatrix}$ rather than the standard $\begin{bmatrix} 0 & I \\ \Delta & 0 \end{bmatrix}$. This makes it possible to associate a boundary triplet to the wave equation (Section 3) and it turns out that also obtaining previously known results becomes technically simpler with this choice. Using the results obtained for the homogeneous case, we show in Section 4 that the inhomogeneous system presented above is an impedance passive boundary control system.

The general boundary triplet techniques that we develop generalise e.g. [JZ12, Thm 7.2.4] to n -dimensional spatial domains, and they are certainly of independent interest as boundary triplets are still being actively used in the study of PDEs; see e.g. [GG91, DHMdS09, Arl12] and the references therein.

In our analysis of the wave equation, we recover the well-known result that the adjoint of the gradient operator, considered as an unbounded operator from $L^2(\Omega)$ into $L^2(\Omega)^n$, is minus the divergence operator, considered as an unbounded operator from $L^2(\Omega)^n$ into $L^2(\Omega)$. Other work making extensive use of the duality between the divergence and the gradient in the analysis of PDEs is [Tro13, Tro14]; this work suggests that there is potential for extending the approach to certain types of non-linearities at the boundary.

We end the introduction with a summary of the structure of the paper. Section 2 presents results for characterizing boundary conditions that induce contraction semigroups, assuming the existence of a boundary triplet. In Section 3, we associate a boundary triplet to the wave equation and show how the results of Section 2 can be applied in this case. Section 4 concerns the interpretation of the wave system as a conservative boundary control

system in different ways: with different choices of input/output spaces, and passivity is considered in both the impedance and scattering sense. The paper also contains two appendices, one with Sobolev-space background and one with two general operator-theoretical results. To our knowledge, Theorem A.8 is new.

2 General results for boundary triplets

We begin by adapting the definition [GG91, p. 155] of a boundary triplet for a symmetric operator to the case of a skew-symmetric operator; see also [MS07, §5].

Definition 2.1. Let A_0 be a densely defined, skew-symmetric, and closed linear operator on a Hilbert space X . By a *boundary triplet* for A_0^* we mean a triple $(\mathcal{B}; B_1, B_2)$ consisting of a Hilbert space \mathcal{B} and two bounded linear operators $B_1, B_2 : \text{dom}(A_0^*) \rightarrow \mathcal{B}$, such that $\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \text{dom}(A_0^*) = \begin{bmatrix} \mathcal{B} \\ \mathcal{B} \end{bmatrix}$ and for all $x, \tilde{x} \in \text{dom}(A_0^*)$ there holds

$$\langle A_0^* x, \tilde{x} \rangle_X + \langle x, A_0^* \tilde{x} \rangle_X = \langle B_1 x, B_2 \tilde{x} \rangle_{\mathcal{B}} + \langle B_2 x, B_1 \tilde{x} \rangle_{\mathcal{B}}. \quad (2.1)$$

Indeed, the analogue of (2.1) is written as follows in [GG91, p. 155]:

$$\langle \mathcal{A}^* x, \tilde{x} \rangle - \langle x, \mathcal{A}^* \tilde{x} \rangle = \langle \Gamma_1 x, \Gamma_2 \tilde{x} \rangle - \langle \Gamma_2 x, \Gamma_1 \tilde{x} \rangle,$$

and setting $A_0^* = (i\mathcal{A})^*$, $B_1 = \Gamma_1$, and $B_2 = i\Gamma_2$ in (2.1), we obtain exactly this. From the definition of boundary triplet it immediately follows that the so-called *minimal operator* A_0 can be recovered via $A_0 = -A_0^*|_{\ker(B_1) \cap \ker(B_2)}$; see [GG91, p. 155].

Let X be a Hilbert space and let R be a (linear) relation in X , i.e., a subspace of X^2 . Then R is called *dissipative* if $\text{Re} \langle r_1, r_2 \rangle_X \leq 0$ for all $\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \in R$, and R is *maximal dissipative* if R has no proper extension to a dissipative relation in X . The relation R is called *skew-symmetric* if $\text{Re} \langle r_1, r_2 \rangle = 0$ for all $\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \in R$, and it is *(maximal) accretive* if $\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} R$ is (maximal) dissipative. An operator $A : X \supset \text{dom}(A) \rightarrow X$ is called *(maximal) dissipative*, *(maximal) accretive*, or *skew-symmetric* if its graph $\mathcal{G}(A) := \begin{bmatrix} I \\ A \end{bmatrix} \text{dom}(A)$, seen as a relation in X , has the corresponding property.

Theorem 2.2. Let $(\mathcal{B}; B_1, B_2)$ be a boundary triplet for A_0^* and consider the restriction A of A_0^* to a subspace \mathcal{D} containing $\ker(B_1) \cap \ker(B_2)$. Define a subspace of \mathcal{B}^2 by $\mathcal{C} := \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \mathcal{D}$. Then the following claims are true:

1. The domain of A can be written

$$\text{dom}(A) = \mathcal{D} = \left\{ d \in \text{dom}(A_0^*) \mid \begin{bmatrix} B_1 d \\ B_2 d \end{bmatrix} \in \mathcal{C} \right\}. \quad (2.2)$$

2. The operator closure of A is A_0^* restricted to

$$\tilde{\mathcal{D}} = \left\{ d \in \text{dom}(A_0^*) \mid \begin{bmatrix} B_1 d \\ B_2 d \end{bmatrix} \in \overline{\mathcal{C}} \right\},$$

where $\overline{\mathcal{C}}$ is the closure of \mathcal{C} in \mathcal{B}^2 . Actually, $\tilde{\mathcal{D}}$ is the closure of \mathcal{D} in $\text{dom}(A_0^*)$, where $\text{dom}(A_0^*)$ is endowed with the graph norm. Furthermore, A is closed if and only if \mathcal{C} is closed.

3. The adjoint A^* is the restriction of $-A_0^*$ to \mathcal{D}' , where

$$\mathcal{D}' = \left\{ d' \in \text{dom}(A_0^*) \mid \begin{bmatrix} B_1 d' \\ B_2 d' \end{bmatrix} \in \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \mathcal{C}^\perp \right\}.$$

4. The operator A is (maximal) dissipative if and only if \mathcal{C} is a (maximal) dissipative relation in \mathcal{B} . Moreover, A is maximal dissipative if and only if there exists a contraction V on \mathcal{B} such that $\mathcal{C} = \ker \begin{pmatrix} I + V & I - V \end{pmatrix}$.

5. The operator A is skew-adjoint if and only if $\mathcal{C} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \mathcal{C}^\perp$. This holds if and only if $\mathcal{C} = \ker \begin{pmatrix} I + V & I - V \end{pmatrix}$ for some unitary operator V on \mathcal{B} .

It also holds that A is (maximal) accretive if and only if \mathcal{C} is (maximal) accretive. Consequently, A is skew-symmetric if and only if \mathcal{C} is skew-symmetric.

In Theorem 2.2, we use the operator A to define a relation \mathcal{C} , but we can also go the other way around: If we start with an arbitrary $\mathcal{C} \subset \mathcal{B}^2$ and define A as the restriction of A_0^* to $\text{dom}(A)$ by the right hand-side in (2.2), then by the surjectivity of $\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, we have $\mathcal{C} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \text{dom}(A)$, and hence all statements in the theorem remain true. Similarly, it follows from part (3) that $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \mathcal{C}^\perp = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \text{dom}(A^*)$. It is thus shown how to obtain \mathcal{C} from $\text{dom}(A)$ and vice versa; part (4) also contains a formula that expresses \mathcal{C} in terms of V . Conversely, we can recover V from \mathcal{C} as the mapping

$$V : e - f \mapsto e + f, \quad \begin{bmatrix} f \\ e \end{bmatrix} \in \mathcal{C}, \quad \text{dom}(V) = \begin{bmatrix} -I & I \end{bmatrix} \mathcal{C}.$$

Indeed, if \mathcal{C} is a maximal dissipative relation in \mathcal{B} , then V defined by this formula is a contraction on \mathcal{B} ; see also Lemma 2.4 below.

Proof. 1. Denote the set on the right-hand side of (2.2) by $\widehat{\mathcal{D}}$. Then by the definition of \mathcal{C} :

$$d \in \mathcal{D} \implies \begin{bmatrix} B_1 d \\ B_2 d \end{bmatrix} \in \mathcal{C} \implies d \in \widehat{\mathcal{D}}.$$

Conversely by the definitions of $\widehat{\mathcal{D}}$ and \mathcal{C} , respectively,

$$\begin{aligned} d \in \widehat{\mathcal{D}} &\implies \begin{bmatrix} B_1 d \\ B_2 d \end{bmatrix} \in \mathcal{C} \implies \exists d' \in \mathcal{D} : \begin{bmatrix} B_1 d \\ B_2 d \end{bmatrix} = \begin{bmatrix} B_1 d' \\ B_2 d' \end{bmatrix} \\ &\implies \exists d' \in \mathcal{D} : d - d' \in \ker \left(\begin{bmatrix} B_1 d \\ B_2 d \end{bmatrix} \right) \subset \mathcal{D}, \end{aligned}$$

and for such a d' we have $d = d - d' + d' \in \mathcal{D}$. Thus $\mathcal{D} = \widehat{\mathcal{D}}$.

2. It follows from Lemma B.1 that $\overline{\text{dom}(A)} = \overline{\text{dom}(A)} = \overline{\mathcal{D}}$; hence A is a closed operator if and only if \mathcal{D} is a closed subspace of $\text{dom}(A_0^*)$. Moreover, by (2.2) and statement (2) of Lemma B.2, we have that $\overline{\mathcal{D}} = \mathcal{D}$ and that \mathcal{D} is closed if and only if \mathcal{C} is closed.

3. From $A \subset A_0^*$ and the definition of the minimal operator, we get $-A_0 \subset A$ which in turn implies that $A^* \subset -A_0^*$. Then it follows from (2.1) that $d' \in \text{dom}(A^*)$ if and only if $\begin{bmatrix} B_2 d' \\ B_1 d' \end{bmatrix} \perp \begin{bmatrix} B_1 d \\ B_2 d \end{bmatrix}$ for all $d \in \mathcal{D}$, and this proves assertion 3.

4. Both claims follow from [GG91, Thm 3.1.6] and its proof.

5. Since $-A^*, A \subset A_0^*$, it holds that $A^* = -A$ if and only if $\text{dom}(A^*) = \text{dom}(A)$. By item 3 and (2.2), $\text{dom}(A^*) = \text{dom}(A)$ if $\mathcal{C} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \mathcal{C}^\perp$. Conversely, if $\text{dom}(A^*) = \text{dom}(A)$, then by the above formulas connecting $\text{dom}(A)$, \mathcal{C} , and $\text{dom}(A^*)$:

$$\mathcal{C} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \text{dom}(A) = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \text{dom}(A^*) = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \mathcal{C}^\perp.$$

The other assertion is contained in [GG91, Thm 3.1.6]. \square

Motivated by item (4) of Theorem 2.2, we now specialise Theorem 2.2 to the case where \mathcal{C} is the kernel of some $W_B \in \mathcal{L}(\mathcal{B}^2; \mathcal{K})$, i.e., W_B is a bounded and everywhere-defined linear operator from \mathcal{B}^2 into \mathcal{K} .

Theorem 2.3. *Let $(\mathcal{B}; B_1, B_2)$ be a boundary triplet for the operator A_0^* on a Hilbert space X , let \mathcal{K} be a Hilbert space, and let $W_B = \begin{bmatrix} W_1 & W_2 \end{bmatrix} \in \mathcal{L}(\mathcal{B}^2; \mathcal{K})$. The following claims are true for the restriction $A := A_0^*|_{\text{dom}(A)}$ to $\text{dom}(A) = \ker \left(\begin{bmatrix} W_1 & W_2 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \right)$:*

1. *The operator A is closed.*
2. *The operator A is (maximal) dissipative if and only if $\ker(W_B)$ is a (maximal) dissipative relation in \mathcal{B} .*
3. *The adjoint of A is $A^* = -A_0^*|_{\text{dom}(A^*)}$, where*

$$\text{dom}(A^*) = \left\{ x \in \text{dom}(A_0^*) \mid \begin{bmatrix} B_1 x \\ B_2 x \end{bmatrix} \in \overline{\text{ran} \left(\begin{bmatrix} W_2^* \\ W_1^* \end{bmatrix} \right)} \right\}. \quad (2.3)$$

4. The adjoint A^* is dissipative if and only if

$$W_1 W_2^* + W_2 W_1^* \geq 0 \quad (2.4)$$

in \mathcal{K} . The adjoint is skew-symmetric, i.e., $\operatorname{Re} \langle A^* x, x \rangle = 0$ for all $x \in \operatorname{dom}(A^*)$, if and only if (2.4) holds with equality.

5. The operator A generates a contraction semigroup on X if and only if A is dissipative and (2.4) holds.

6. The operator A generates a unitary group on X if and only if A is skew-symmetric and (2.4) holds with equality.

Proof. The subspace \mathcal{D} of Theorem 2.2 is

$$\mathcal{D} = \operatorname{dom}(A) = \ker \left(\begin{bmatrix} W_1 & W_2 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \right) \supset \ker(B_1) \cap \ker(B_2).$$

By (2.2) and the surjectivity of $\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, it is easy to see that $\mathcal{C} = \ker(W_B)$.

1. Since $W_B \in \mathcal{L}(\mathcal{B}^2; \mathcal{K})$, $\mathcal{C} = \ker(W_B)$ is closed. Now the closedness of $\operatorname{dom}(A)$ follows from part (2) of Theorem 2.2.

2. This follows from $\mathcal{C} = \ker(W_B)$ and part (4) of Theorem 2.2.

3. The domain and action of A^* follow directly from part (3) of Theorem 2.2; note that

$$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \mathcal{C}^\perp = \overline{\operatorname{ran} \left(\begin{bmatrix} W_2^* \\ W_1^* \end{bmatrix} \right)}. \quad (2.5)$$

4. Applying Theorem 2.2 to A^* , using (2.5), we obtain that A^* is dissipative if and only if $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \mathcal{C}^\perp$ is accretive; note the minus sign in the formula for A^* in item 3. By the continuity of the inner product this holds if and only if $\operatorname{ran} \left(\begin{bmatrix} W_2^* \\ W_1^* \end{bmatrix} \right)$ is accretive, but this is true if and only if (2.4) holds, since

$$2\operatorname{Re} \langle W_2^* f, W_1^* f \rangle_{\mathcal{B}} = \langle (W_1 W_2^* + W_2 W_1^*) f, f \rangle_{\mathcal{K}}, \quad f \in \mathcal{K}.$$

A trivial modification of the above gives the proof for the skew-symmetric case.

5. Since A is closed by the first item, this follows from the Lumer-Phillips Theorem.

6. Since A is closed, both A and A^* are skew-symmetric if and only if and only if $A^* = -A$. The claim follows from Stone's theorem. \square

We next introduce a maximality condition, which implies that A is dissipative if and only if A^* is dissipative. Theorem 2.5 is a general boundary triplet analogue of [JZ12, Thm 7.2.4]. This theorem can be applied to some PDEs on n -dimensional spatial domains to show existence of solutions; see also [GZM05, §4.1]. First, however, we need the following lemma:

Lemma 2.4. *Let \mathcal{B} and \mathcal{K} be Hilbert spaces, and let $\begin{bmatrix} W_1 & W_2 \end{bmatrix} \in \mathcal{L}(\mathcal{B}^2; \mathcal{K})$. Assume that $W_1 + W_2$ is injective, and that*

$$\operatorname{ran}(W_1 - W_2) \subset \operatorname{ran}(W_1 + W_2). \quad (2.6)$$

Then there exists a unique $V \in \mathcal{L}(\mathcal{B})$ such that

$$(W_1 + W_2)V = W_1 - W_2, \quad (2.7)$$

or equivalently,

$$\begin{bmatrix} W_1 & W_2 \end{bmatrix} = \frac{1}{2}(W_1 + W_2) \begin{bmatrix} I + V & I - V \end{bmatrix}. \quad (2.8)$$

Hence, $\ker \left(\begin{bmatrix} W_1 & W_2 \end{bmatrix} \right) = \ker \left(\begin{bmatrix} I + V & I - V \end{bmatrix} \right)$ and, moreover, the operator inequality $W_1 W_2^ + W_2 W_1^* \geq 0$ holds in \mathcal{K} if and only if $V V^* \leq I$ in \mathcal{B} .*

We point out that $W_1 W_2^* + W_2 W_1^* \geq 0$ can equivalently be written as

$$\begin{bmatrix} W_1 & W_2 \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} W_1 & W_2 \end{bmatrix}^* \geq 0. \quad (2.9)$$

Proof. We first establish the existence and uniqueness of a $V \in \mathcal{L}(\mathcal{B})$ such that (2.7) holds. Since $W_1 + W_2$ is injective, there exists a closed left inverse $(W_1 + W_2)^{-l}$ defined on $\operatorname{ran}(W_1 + W_2) \oplus (\operatorname{ran}(W_1 + W_2))^\perp$. Defining

$$V := (W_1 + W_2)^{-l}(W_1 - W_2),$$

we obtain from (2.6) that V is defined on all of \mathcal{B} . By the boundedness of $W_1 - W_2$ and the closedness of $(W_1 + W_2)^{-l}$, the composition V is closed, and hence $V \in \mathcal{B}$ by the closed graph theorem. Using assumption (2.6), for all $b \in \mathcal{B}$ there exists a $z \in \mathcal{B}$ such that $(W_1 - W_2)b = (W_1 + W_2)z$, and we obtain (2.7):

$$\begin{aligned} (W_1 + W_2)Vb &= (W_1 + W_2)(W_1 + W_2)^{-l}(W_1 - W_2)b \\ &= (W_1 + W_2)(W_1 + W_2)^{-l}(W_1 + W_2)z \\ &= (W_1 + W_2)z = (W_1 - W_2)b. \end{aligned}$$

On the other hand, because of the injectivity of $W_1 + W_2$, the operator V is uniquely determined by (2.7).

Now assume that $W_1 W_2^* + W_2 W_1^* \geq 0$; we prove that V is a contraction. First note that

$$(W_1 - W_2)^* \left((W_1 + W_2)^{-l} \right)^* \subset V^*,$$

where the left-hand side is defined densely in \mathcal{B} since $(W_1 + W_2)^{-l}$ is densely defined and $(W_1 - W_2)^* \in \mathcal{L}(\mathcal{K}; \mathcal{B})$; hence it suffices to show that $(W_1 -$

$W_2)^* ((W_1 + W_2)^{-l})^*$ is contractive. As \mathcal{B} and \mathcal{K} are Hilbert spaces and W_1, W_2 bounded, we have that $W_1 W_2^* + W_2 W_1^* \geq 0$ is equivalent to

$$\|(W_1 - W_2)^* x\|^2 \leq \|(W_1 + W_2)^* x\|^2, \quad x \in \mathcal{B}. \quad (2.10)$$

For arbitrary $y \in \text{dom} \left(((W_1 + W_2)^{-l})^* \right)$, we set $x := ((W_1 + W_2)^{-l})^* y$ and obtain from (2.10) that $\|(W_1 - W_2)^* ((W_1 + W_2)^{-l})^* y\|^2 \leq \|y\|^2$. We conclude that $W_1 W_2^* + W_2 W_1^* \geq 0$ implies that $V V^* \leq I$. Conversely, if $V V^* \leq I$, then using (2.8) in (2.9), we have

$$\begin{bmatrix} W_1 & W_2 \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} W_1 & W_2 \end{bmatrix}^* = \frac{1}{2} (W_1 + W_2) (I - V V^*) (W_1 + W_2)^* \geq 0.$$

Finally, it is straightforward to verify that (2.7) is equivalent to (2.8); the equality of the kernels then follows from the injectivity of $W_1 + W_2$. \square

If $W_1 + W_2 : \mathcal{B} \rightarrow \mathcal{K}$ is invertible then (2.6) holds. A good choice of \mathcal{K} can sometimes make this possible.

Theorem 2.5. *Let A and $\begin{bmatrix} W_1 & W_2 \end{bmatrix}$ be the operators in Theorem 2.3, and assume that (2.6) holds. Then the following conditions are equivalent:*

1. *The operator A generates a contraction semigroup on X .*
2. *The operator A is dissipative.*
3. *The operator $W_1 + W_2$ is injective and the following operator inequality holds in \mathcal{K} :*

$$W_1 W_2^* + W_2 W_1^* \geq 0. \quad (2.11)$$

Proof. The Lumer-Phillips Theorem provides the *implication from 1 to 2*.

We now prove that *assertion 2 implies assertion 3*. By part (2) of Theorem 2.3 we know that $\mathcal{C} = \ker(W_B)$ is dissipative. So for every $\begin{bmatrix} h \\ k \end{bmatrix} \in \ker(W_B)$ there holds

$$\text{Re} \langle h, k \rangle_{\mathcal{B}} \leq 0. \quad (2.12)$$

If $y \in \ker(W_1 + W_2)$, then $W_B \begin{bmatrix} y \\ y \end{bmatrix} = 0$ and by (2.12), $\text{Re} \|y\|_{\mathcal{B}}^2 \leq 0$. Thus $y = 0$ and $W_1 + W_2$ is injective. By Lemma 2.4, there exists a $V \in \mathcal{L}(\mathcal{B})$, such that (2.8) holds and $\ker(W_B) = \begin{bmatrix} I + V & I - V \end{bmatrix}$.

Let $u \in \mathcal{B}$ be arbitrary and set $y := Vu$. Then $\begin{bmatrix} y - u \\ y + u \end{bmatrix}$ lies in the dissipative $\ker \left(\begin{bmatrix} I + V & I - V \end{bmatrix} \right)$ and hence $\|Vu\|^2 - \|u\|^2 = \text{Re} \langle y - u, y + u \rangle \leq 0$, which proves that V is a contraction. Lemma 2.4 gives that (2.11) holds.

Assertion 3 implies assertion 1. The assumptions of Lemma 2.4 are satisfied and in addition (2.11) holds, and so there exists a contraction V on \mathcal{B} satisfying (2.8). By part (4) of Theorem 2.2, A_0^* restricted to $\mathcal{D} := \{d \in \text{dom}(A_0^*) \mid \begin{bmatrix} B_1 d \\ B_2 d \end{bmatrix} \in \ker([I + V, I - V])\}$ is maximal dissipative and thus the

infinitesimal generator of a contraction semigroup. From the injectivity of $W_1 + W_2$ and (2.8), we see that \mathcal{D} equals $\{d \in \text{dom}(A_0^*) \mid \begin{bmatrix} B_1 d \\ B_2 d \end{bmatrix} \in \ker(W_B)\}$. Thus A generates a contraction semigroup. \square

The boundary triplet that we shall associate to the wave equation in the next section is of the “pivoted” type described in the following result, which is also important in the proof of Theorem 4.4 below.

Theorem 2.6. *Let \mathcal{B} be a Hilbert space densely and continuously contained in a Hilbert space \mathcal{B}_0 , let \mathcal{B}' be the dual of \mathcal{B} with pivot space \mathcal{B}_0 , and let $\Psi : \mathcal{B}' \rightarrow \mathcal{B}$ be a unitary operator. Let b_2 be a bounded operator from $\text{dom}(A_0^*)$ to \mathcal{B}' and assume that $(\mathcal{B}; B_1, \Psi b_2)$ is a boundary triplet for the operator A_0^* on the Hilbert space X .*

Let $V_B = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \in \mathcal{L}(\mathcal{B}_0^2; \mathcal{K})$, where \mathcal{K} is some Hilbert space, and define

$$\mathcal{A} := \left\{ a \in \text{dom}(A_0^*) \mid b_2 a \in \mathcal{B}_0 \wedge \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} B_1 \\ b_2 \end{bmatrix} a = 0 \right\}. \quad (2.13)$$

Then the following two conditions are together sufficient for the closure A of the operator $A_0^|_{\mathcal{A}}$ (closure in the sense of an operator on X) to generate a contraction semigroup on X :*

1. *$\text{Re } \langle u, v \rangle_{\mathcal{B}_0} \leq 0$ for all $u, v \in \mathcal{B}_0$ such that $V_1 u + V_2 v = 0$.*
2. *The following operator inequality holds in \mathcal{K} :*

$$V_1 V_2^* + V_2 V_1^* \geq 0. \quad (2.14)$$

The operator A generates a unitary group if $\text{Re } \langle u, v \rangle_{\mathcal{B}_0} = 0$ for all $\begin{bmatrix} u \\ v \end{bmatrix} \in \ker(V_B)$ and $V_1 V_2^ + V_2 V_1^* = 0$.*

Condition 2 is also necessary for A to generate a contraction semigroup (unitary group) on X .

Here we have changed to a small b in the boundary mapping b_2 in order to avoid confusion. The mapping B_2 used previously is analogous to Ψb_2 here. If one wanted to try to reduce Theorem 2.6 to Theorem 2.3, then one might try to set $W_1 := V_1|_{\mathcal{B}}$ and $W_2 := V_2 \Psi^*$. However, this does not go through without complications, because $V_2 \Psi^*$ is in general defined only on $\Psi \mathcal{B}_0$, and not bounded from \mathcal{B} into \mathcal{K} .

Proof. By the definition of A , \mathcal{A} is dense in $\text{dom}(A)$, and so the closed operator A is maximal dissipative if and only if $A|_{\mathcal{A}}$ is dissipative and $A|_{\mathcal{A}}^* = A^*$ is dissipative.

By (2.13), we have

$$\begin{aligned}
a \in \mathcal{A} &\iff \begin{bmatrix} B_1 \\ b_2 \end{bmatrix} a \in \begin{bmatrix} \mathcal{B} \\ \mathcal{B}_0 \end{bmatrix} \cap \ker \begin{pmatrix} V_1 & V_2 \end{pmatrix} \\
&\iff \begin{bmatrix} B_1 \\ \Psi b_2 \end{bmatrix} a \in \begin{bmatrix} \mathcal{B} \\ \Psi \mathcal{B}_0 \end{bmatrix} \cap \ker \begin{pmatrix} V_1 & V_2 \Psi^* \end{pmatrix} \\
&\iff \begin{bmatrix} B_1 \\ \Psi b_2 \end{bmatrix} a \in \ker \left(\begin{bmatrix} V_1|_{\mathcal{B}} & V_2 \Psi^*|_{\Psi \mathcal{B}_0} \end{bmatrix} \right).
\end{aligned} \tag{2.15}$$

From this we see that the space \mathcal{C} in Theorem 2.2 is

$$\mathcal{C} = \left\{ \begin{bmatrix} q \\ p \end{bmatrix} \in \mathcal{B}^2 \mid \exists \tilde{p} \in \mathcal{B}_0 : \quad p = \Psi \tilde{p}, \quad V_1 q + V_2 \Psi^* p = 0 \right\}. \tag{2.16}$$

For $\begin{bmatrix} q \\ p \end{bmatrix} \in \mathcal{C}$ there holds

$$\langle q, p \rangle_{\mathcal{B}} = \langle q, \Psi \tilde{p} \rangle_{\mathcal{B}} = \langle q, \tilde{p} \rangle_{\mathcal{B}, \mathcal{B}'} = \langle q, \tilde{p} \rangle_{\mathcal{B}_0} \leq 0,$$

where we used condition 1. Theorem 2.2 now yields that $A_0^*|_{\mathcal{A}}$ is dissipative, and by the continuity of the inner product A is also dissipative. The same argument gives that A is skew-symmetric in case $V_1 u + V_2 v = 0$ implies that $\operatorname{Re} \langle u, v \rangle = 0$.

We next calculate A^* and verify that this adjoint is dissipative if and only if (2.14) holds. By items 1–3 in Theorem 2.2, the denseness of \mathcal{A} in $\operatorname{dom}(A)$, and (2.15), we obtain

$$\begin{aligned}
d \in \operatorname{dom}(A^*) &\iff \begin{bmatrix} \Psi b_2 d \\ B_1 d \end{bmatrix} \in \mathcal{B}^2 \ominus \left(\begin{bmatrix} B_1 \\ \Psi b_2 \end{bmatrix} \mathcal{A} \right) \\
&\iff \begin{bmatrix} \Psi b_2 d \\ B_1 d \end{bmatrix} \in \overline{\operatorname{ran} \left(\begin{bmatrix} (V_1|_{\mathcal{B}})^{\dagger} \\ (V_2 \Psi^*|_{\Psi \mathcal{B}_0})^{\dagger} \end{bmatrix} \right)}^{\mathcal{B}^2},
\end{aligned}$$

where \dagger denotes the adjoint calculated with respect to the inner product in \mathcal{B} instead of that in \mathcal{B}_0 . Since $A^* = -A_0^*|_{\operatorname{dom}(A^*)}$, we obtain from part (4) of Theorem 2.2 that A^* is dissipative if and only if $\operatorname{ran} \left(\begin{bmatrix} (V_2 \Psi^*|_{\Psi \mathcal{B}_0})^{\dagger} \\ (V_1|_{\mathcal{B}})^{\dagger} \end{bmatrix} \right)$ is an accretive relation in \mathcal{B} . We finish the proof by verifying that this is indeed the case, assuming (2.14).

It holds that

$$\langle V_1 u, k \rangle_{\mathcal{K}} = \langle u, V_1^* k \rangle_{\mathcal{B}_0} = \langle u, V_1^* k \rangle_{\mathcal{B}, \mathcal{B}'} = \langle u, \Psi V_1^* k \rangle_{\mathcal{B}}$$

for all $u \in \mathcal{B}$ and $k \in \mathcal{B}_0$, and thus $(V_1|_{\mathcal{B}})^{\dagger} = \Psi V_1^*$. Moreover, $k \in \operatorname{dom} \left((V_2 \Psi^*|_{\Psi \mathcal{B}_0})^{\dagger} \right)$ if and only if there exists some $s \in \mathcal{B}$ such that

$$\langle V_2 \Psi^* v, k \rangle_{\mathcal{K}} = \langle v, s \rangle_{\mathcal{B}}, \quad v \in \Psi \mathcal{B}_0. \tag{2.17}$$

Now assume that $k \in \text{dom} \left((V_2 \Psi^*|_{\Psi \mathcal{B}_0})^\dagger \right)$ and choose a $s \in \mathcal{B}$ such that (2.17) holds. Then it holds for all $v \in \Psi \mathcal{B}_0$ that

$$\langle \Psi^* v, s \rangle_{\mathcal{B}_0} = (\Psi^* v, s)_{\mathcal{B}', \mathcal{B}} = \langle v, s \rangle_{\mathcal{B}} = \langle V_2 \Psi^* v, k \rangle_{\mathcal{K}} = \langle \Psi^* v, V_2^* k \rangle_{\mathcal{B}_0},$$

i.e., that $V_2^* k = s \in \mathcal{B}$. Conversely, $V_2^* k \in \mathcal{B}$ implies that $k \in \text{dom} \left((V_2 \Psi^*|_{\Psi \mathcal{B}_0})^\dagger \right)$, because then we obtain for all $v \in \Psi \mathcal{B}_0$ that

$$\langle v, V_2^* k \rangle_{\mathcal{B}} = (\Psi^* v, V_2^* k)_{\mathcal{B}', \mathcal{B}} = \langle \Psi^* v, V_2^* k \rangle_{\mathcal{B}_0} = \langle V_2 \Psi^* v, k \rangle_{\mathcal{K}}.$$

We conclude that $(V_2 \Psi^*|_{\Psi \mathcal{B}_0})^\dagger$ is the restriction of V_2^* to

$$\text{dom} \left((V_2 \Psi^*|_{\Psi \mathcal{B}_0})^\dagger \right) = \{k \in \mathcal{K} \mid V_2^* k \in \mathcal{B}\}.$$

By definition

$$\text{ran} \left(\left[\begin{array}{c} (V_2 \Psi^*|_{\Psi \mathcal{B}_0})^\dagger \\ (V_1|_{\mathcal{B}})^\dagger \end{array} \right] \right) = \left\{ \left[\begin{array}{c} V_2^* k \\ \Psi V_1^* k \end{array} \right] \mid k \in \mathcal{K} \wedge V_2^* k \in \mathcal{B} \right\}$$

is an accretive relation in \mathcal{B} if and only if for all $k \in \mathcal{K}$ with $V_2^* k \in \mathcal{B}$ it holds that

$$\text{Re} \langle V_2^* k, \Psi V_1^* k \rangle_{\mathcal{B}} = \text{Re} (V_2^* k, V_1^* k)_{\mathcal{B}, \mathcal{B}'} = \text{Re} \langle V_2^* k, V_1^* k \rangle_{\mathcal{B}_0} = \text{Re} \langle V_1 V_2^* k, k \rangle_{\mathcal{K}} \geq 0.$$

This is clearly true if (2.14) holds. Conversely, if the relation is accretive, then (2.14) holds, since $\{k \in \mathcal{K} \mid V_2^* k \in \mathcal{B}\} = \text{dom} \left((V_2 \Psi^*|_{\Psi \mathcal{B}_0})^\dagger \right)$ which is dense in \mathcal{K} . \square

We end the section with the following remark: The only implication in the proof of Theorem 2.6, which is not an equivalence, is where dissipativity of $\begin{bmatrix} V_1 & V_2 \end{bmatrix}$ implies dissipativity of $\begin{bmatrix} \mathcal{B} \\ \mathcal{B}_0 \end{bmatrix} \cap \ker \left(\begin{bmatrix} V_1 & V_2 \end{bmatrix} \right)$. If the intersection is dense in $\ker \left(\begin{bmatrix} V_1 & V_2 \end{bmatrix} \right)$, then the converse implication is also true by the continuity of the inner product. In this case Theorem 2.6 gives necessary and sufficient conditions for A to generate a contraction semigroup.

3 The wave equation

The notation of this section is described in Appendix A, with the additional observation that Γ_\bullet in the appendix equals $\Gamma_1 \cup \Gamma_2$ in this section. In the rest of the article, we throughout assume that $\Omega \subset \mathbb{R}^n$ is a bounded set with Lipschitz-continuous boundary $\partial\Omega$. It is moreover convenient for us to introduce the concept of a “splitting with thin common boundary”:

Definition 3.1. By a *splitting of $\partial\Omega$ with thin boundaries*, we mean a finite collection of subsets $\Gamma_k \subset \partial\Omega$, such that:

1. $\bigcup_k \overline{\Gamma_k} = \partial\Omega$,
2. the sets Γ_k are pairwise disjoint,
3. the sets Γ_k are open in the relative topology of $\partial\Omega$, and
4. the boundaries of the sets Γ_k all have surface measure zero.

For instance, if the subset Γ_k has Lipschitz-continuous boundary, then the surface measure of $\partial\Gamma_k$ is zero. In the sequel, we always assume the boundary $\partial\Omega$ to be split into subsets with thin boundaries. If we furthermore regard $L^2(\Pi)$, $\Pi \subset \partial\Omega$, as the space of $f \in L^2(\partial\Omega)$ that satisfy $f(x) = 0$ for almost every $x \in \partial\Omega \setminus \Pi$, then it holds that

$$L^2(\partial\Omega) = \bigoplus_k L^2(\Gamma_k)$$

and we denote the corresponding orthogonal projections by π_k . If $\{\Gamma_0, \Gamma_1\}$ is a splitting of $\partial\Omega$ with thin boundaries, then

$$L^2(\partial\Omega) = L^2(\Gamma_0) \oplus L^2(\Gamma_1) \oplus L^2(\partial\Omega \setminus (\Gamma_0 \cup \Gamma_1)) = L^2(\Gamma_0) \oplus L^2(\Gamma_1),$$

since $\partial\Omega \setminus (\Gamma_0 \cup \Gamma_1) = \partial\Gamma_0 \cup \partial\Gamma_1$ has zero surface measure; see also [TW09, p. 427].

The rest of the paper is devoted to a study of the wave equation (1.1). We will recall the definitions of scattering and impedance passive and conservative boundary control systems. We shall also associate two impedance passive boundary control systems to (1.1) using different input- and output spaces; the flavour is similar to [MS07, §6.2]. The main step of the proof is an application of Theorem 2.6 to show that (1.1) is governed by a contraction semigroup on $L^2(\Omega)^{n+1}$ equipped with a modified but equivalent norm.

For physical reasons the mass density $\rho(\cdot) \in L^\infty(\Omega)$ takes real positive values and Young's modulus $T(\cdot) \in L^\infty(\Omega)^{n \times n}$ satisfies $T(\xi)^* = T(\xi)$ for almost all $\xi \in \Omega$. We make the additional (physically reasonable) assumption that there exists a $\delta > 0$, such that $\rho(\xi) \geq \delta$, and $T(\xi) \geq \delta I$ for almost all $\xi \in \Omega$. We let Q_i and Q_b be bounded and accretive operators on $L^2(\Omega)$ and $L^2(\Gamma_1)$, respectively. If damping inside Ω is absent, then $Q_i = 0$, and if there is no damping at the boundary, then $\Gamma_1 = \emptyset$.

The assumptions we made on the parameters imply that the following multiplication operator is bounded, self-adjoint, and uniformly accretive on $\begin{bmatrix} L^2(\Omega) \\ L^2(\Omega)^n \end{bmatrix}$:

$$\mathcal{H}x := \xi \mapsto \begin{bmatrix} 1/\rho(\xi) & 0 \\ 0 & T(\xi) \end{bmatrix} x(\xi), \quad \xi \in \Omega, \quad x \in \begin{bmatrix} L^2(\Omega) \\ L^2(\Omega)^n \end{bmatrix}. \quad (3.1)$$

Hence this operator defines an alternative, but equivalent, inner product on $\begin{bmatrix} L^2(\Omega) \\ L^2(\Omega)^n \end{bmatrix}$ through $\langle z_1, z_2 \rangle_{\mathcal{H}} := \langle \mathcal{H}z_1, z_2 \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\begin{bmatrix} L^2(\Omega) \\ L^2(\Omega)^n \end{bmatrix}$. We denote $\begin{bmatrix} L^2(\Omega) \\ L^2(\Omega)^n \end{bmatrix}$ equipped with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ by $\mathcal{X}_{\mathcal{H}}$.

We invite the reader to carry out the straightforward verification that the first two lines of the PDE (1.1) correspond to the following abstract ordinary differential equation

$$\dot{x}(t) = (S - Q) \mathcal{H}x(t), \quad t \geq 0, \quad (3.2)$$

where the dot denotes derivative with respect to time, the state vector $x(t) = \begin{bmatrix} M_{\rho} \dot{z}(t) \\ \text{grad } z(t) \end{bmatrix}$ consists of the infinitesimal momentum and strain at the point $\xi \in \Omega$, M_{ρ} is the operator in $L^2(\Omega)$ of multiplication by ρ ,

$$S = \begin{bmatrix} 0 & \text{div} \\ \text{grad} & 0 \end{bmatrix} \Big|_{\text{dom}(S)}, \quad \text{dom}(S) = \begin{bmatrix} H_{\Gamma_0}^1(\Omega) \\ H^{\text{div}}(\Omega) \end{bmatrix}, \quad \text{and} \quad Q = \begin{bmatrix} Q_i & 0 \\ 0 & 0 \end{bmatrix}.$$

Note how the boundary condition on line two of (1.1) becomes part of the domain of S . When we initialise (1.1) with the initial conditions $z(\xi, 0) = z_0(\xi)$ and $\dot{z}(\xi, 0) = w_0(\xi)$, $\xi \in \Omega$, then the corresponding initial state for (3.2) will be $x(0) = \begin{bmatrix} M_{\rho} w_0 \\ \text{grad } z_0 \end{bmatrix}$. At this point, any constants in z_0 disappear, but they can be recovered using Theorem 4.5 below.

The operator $-Q\mathcal{H} = \begin{bmatrix} Q_i M_{1/\rho} & 0 \\ 0 & 0 \end{bmatrix}$ in (3.2) is dissipative, bounded and defined on all of $\mathcal{X}_{\mathcal{H}}$. By the passive majoration technique in [ALM13, Thm 3.2], we may without loss of generality assume that $Q_i = 0$ in the sequel.

3.1 A boundary triplet and contraction semigroups

We shall associate a boundary triplet to the wave equation. The main objective is to apply the results in Section 2 in order to characterise boundary conditions giving a contraction semigroup.

In Appendix A, we give the definitions of the Sobolev spaces $H^1(\Omega)$, $H^{\text{div}}(\Omega)$, $H_{\Gamma_0}^1(\Omega)$, and $H^{1/2}(\partial\Omega)$. Moreover, we define the Dirichlet trace $\gamma_0 : H^1(\Omega) \rightarrow H^{1/2}(\partial\Omega)$, which maps $H_{\Gamma_0}^1(\Omega)$ onto $\mathcal{W} \subset L^2(\Gamma_1 \cup \Gamma_2)$. Furthermore, we introduce the restricted normal trace $\gamma_{\perp} : H^{\text{div}}(\Omega) \rightarrow \mathcal{W}'$, where \mathcal{W}' is the dual of \mathcal{W} with pivot space $L^2(\Gamma_1 \cup \Gamma_2)$. Note that γ_{\perp} is *not* a Neumann trace γ_N ; if $\Gamma_0 = \emptyset$, then $\mathcal{W} = H^{1/2}(\partial\Omega)$ and the relation between the two operators is $\gamma_N x = \gamma_{\perp} \text{grad } x$, for x smooth enough, where the equality is in $H^{-1/2}(\partial\Omega)$. Finally define the Hilbert space

$$H_{\Gamma_0}^{\text{div}}(\Omega) := \ker(\gamma_{\perp}) \quad (3.3)$$

with the norm inherited from $H^{\text{div}}(\Omega)$.

We next show how (1.1) is associated to a contraction semigroup on $\mathcal{X}_{\mathcal{H}}$ by setting $u(\cdot, t) = 0$ for all $t \geq 0$ and disregarding the output equation on the last line of (1.1). To this end, we combine line four (with $u = 0$) and line three of (1.1) by writing

$$\begin{bmatrix} Q_b \pi_1 \\ 0 \end{bmatrix} \gamma_0 \frac{\partial z}{\partial t} + \gamma_{\perp} T \operatorname{grad} z = 0.$$

where π_1 is the orthogonal projection of $L^2(\Gamma_1 \cup \Gamma_2)$ onto $L^2(\Gamma_1)$. More precisely, we shall show the more general statement that the operator

$$A_{\mathcal{H}} := S\mathcal{H}|_{\operatorname{dom}(A_{\mathcal{H}})}, \quad \operatorname{dom}(A_{\mathcal{H}}) := \left\{ x \in \mathcal{H}^{-1} \begin{bmatrix} H_{\Gamma_0}^1(\Omega) \\ H^{\operatorname{div}}(\Omega) \end{bmatrix} \mid [\tilde{Q}_b \gamma_0 \quad \gamma_{\perp}] \mathcal{H}x = 0 \right\}, \quad (3.4)$$

generates a contraction semigroup on $\mathcal{X}_{\mathcal{H}}$, for an arbitrary accretive $\tilde{Q}_b \in \mathcal{L}(\mathcal{W}; \mathcal{W}')$; note that $\tilde{Q}_b \in \mathcal{L}(L^2(\Gamma_1 \cup \Gamma_2))$ implies that $\tilde{Q}_b \in \mathcal{L}(\mathcal{W}; \mathcal{W}')$.

Theorem 3.2. *Let Ω be a bounded Lipschitz set. The operator*

$$A_0 := \begin{bmatrix} 0 & -\operatorname{div} \\ -\operatorname{grad} & 0 \end{bmatrix} \mathcal{H}, \quad \operatorname{dom}(A_0) := \mathcal{H}^{-1} \begin{bmatrix} H_{\Gamma_0}^1(\Omega) \\ H^{\operatorname{div}}(\Omega) \end{bmatrix},$$

is closed, skew-symmetric, and densely defined on $\mathcal{X}_{\mathcal{H}}$. Its adjoint is

$$A_0^* = \begin{bmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{bmatrix} \mathcal{H}, \quad \operatorname{dom}(A_0^*) = \mathcal{H}^{-1} \begin{bmatrix} H_{\Gamma_0}^1(\Omega) \\ H^{\operatorname{div}}(\Omega) \end{bmatrix}. \quad (3.5)$$

Let $M_{1/\rho}$ and M_T be the multiplication operators on the diagonal of \mathcal{H} in (3.1), and set $B_0 := [\gamma_0 M_{1/\rho} \quad 0]$ and $B_{\perp} := [0 \quad \gamma_{\perp} M_T]$. Then $(\mathcal{W}; B_0, \Psi_{\mathcal{W}} B_{\perp})$ is a boundary triplet for A_0^ , where $\Psi_{\mathcal{W}} : \mathcal{W}' \rightarrow \mathcal{W}$ is any unitary operator. In particular,*

$$\langle A_0^* x, \tilde{x} \rangle_{\mathcal{X}_{\mathcal{H}}} + \langle x, A_0^* \tilde{x} \rangle_{\mathcal{X}_{\mathcal{H}}} = \langle B_0 x, \Psi_{\mathcal{W}} B_{\perp} \tilde{x} \rangle_{\mathcal{W}} + \langle \Psi_{\mathcal{W}} B_{\perp} x, B_0 \tilde{x} \rangle_{\mathcal{W}}, \quad x, \tilde{x} \in \operatorname{dom}(A_0^*). \quad (3.6)$$

Proof. That $\begin{bmatrix} B_0 \\ \Psi_{\mathcal{W}} B_{\perp} \end{bmatrix} \mathcal{H}^{-1} \begin{bmatrix} H_{\Gamma_0}^1(\Omega) \\ H^{\operatorname{div}}(\Omega) \end{bmatrix} = \mathcal{W}^2$ follows from $\gamma_0 H_{\Gamma_0}^1(\Omega) = \mathcal{W}$ and $\gamma_{\perp} H^{\operatorname{div}}(\Omega) = \mathcal{W}'$; see Theorem A.8.

The identity (3.6) is obtained by polarizing the following consequence of the integration by parts formula (A.5): For all $\begin{bmatrix} M_{1/\rho} g \\ M_T f \end{bmatrix} \in \begin{bmatrix} H_{\Gamma_0}^1(\Omega) \\ H^{\operatorname{div}}(\Omega) \end{bmatrix}$, we obtain

$$\begin{aligned} 2\operatorname{Re} \left\langle \begin{bmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{bmatrix} \begin{bmatrix} M_{1/\rho} g \\ M_T f \end{bmatrix}, \begin{bmatrix} M_{1/\rho} g \\ M_T f \end{bmatrix} \right\rangle_{L^2(\Omega)^{n+1}} &= \\ 2\operatorname{Re} \left(\langle \operatorname{div} M_T f, M_{1/\rho} g \rangle_{L^2(\Omega)} + \langle M_T f, \operatorname{grad} M_{1/\rho} g \rangle_{L^2(\Omega)^n} \right) &= \\ 2\operatorname{Re} (\gamma_{\perp} M_T f, \gamma_0 M_{1/\rho} g)_{\mathcal{W}', \mathcal{W}} &= \\ 2\operatorname{Re} \left\langle \Psi_{\mathcal{W}} B_{\perp} \begin{bmatrix} g \\ f \end{bmatrix}, B_0 \begin{bmatrix} g \\ f \end{bmatrix} \right\rangle_{\mathcal{W}}; & \end{aligned} \quad (3.7)$$

in the last equality we also used that $\Psi_{\mathcal{W}}$ is unitary.

Now the adjoint of A_0^* in (3.5) is $-A_0^*$ restricted to $\ker(B_0) \cap \ker(\Psi_{\mathcal{W}}B_{\perp})$. This space equals $\mathcal{H}^{-1} \begin{bmatrix} H_0^1(\Omega) \\ H_{\Gamma_0}^{\text{div}}(\Omega) \end{bmatrix}$ by Lemma A.4 and (3.3), i.e., $(A_0^*)^* = A_0$, so that A_0 is closed and obviously it is also densely defined. Finally A_0 is skew-symmetric by (3.7). \square

An interesting special case is obtained by taking ρ and T identities, in which case $\mathcal{X}_{\mathcal{H}}$ reduces to $L^2(\Omega)^{n+1}$. On the other hand, taking $\Gamma_0 = \emptyset$, we get the following special case:

Corollary 3.3. *The operator $-\begin{bmatrix} 0 & \text{div} \\ \text{grad} & 0 \end{bmatrix} \mathcal{H} \Big|_{\mathcal{H}^{-1} \begin{bmatrix} H_0^1(\Omega) \\ H_0^{\text{div}}(\Omega) \end{bmatrix}}$ is closed, symmetric, and densely defined. A boundary triplet for its adjoint $\begin{bmatrix} 0 & \text{div} \\ \text{grad} & 0 \end{bmatrix} \mathcal{H}$ is given by*

$$\left(H^{1/2}(\partial\Omega); [\gamma_0 M_{1/\rho} \quad 0], \Psi_{1/2} \begin{bmatrix} 0 & \gamma_{\perp} \end{bmatrix} M_T \right),$$

where $\Psi_{1/2} : H^{-1/2}(\partial\Omega) \rightarrow H^{1/2}(\partial\Omega)$ is some arbitrary unitary operator.

The following by-product of Theorem 3.2 gives an exact statement on the duality of the divergence and gradient operators. Surprisingly, we were unable to find a citation of this well-known result.

Corollary 3.4. *Let $\Gamma_0 \subset \partial\Omega$ be open. Then $\text{grad}|_{H_{\Gamma_0}^1(\Omega)}^* = -\text{div}|_{H_{\Gamma_0}^{\text{div}}(\Omega)}$ and $\text{grad}|_{H_0^1(\Omega)}^* = -\text{div}|_{H^{\text{div}}(\Omega)}$.*

Proof. Choosing \mathcal{H} to be the identity in Theorem 3.2, we obtain that A_0 is given by

$$A_0 = \begin{bmatrix} 0 & -\text{div}|_{H_{\Gamma_0}^{\text{div}}(\Omega)} \\ -\text{grad}|_{H_0^1(\Omega)} & 0 \end{bmatrix}.$$

Using this expression and equation (3.5), we find that A_0^* satisfies

$$\begin{bmatrix} 0 & -\text{grad}|_{H_0^1(\Omega)}^* \\ -\text{div}|_{H_{\Gamma_0}^{\text{div}}(\Omega)}^* & 0 \end{bmatrix} = A_0^* = \begin{bmatrix} 0 & \text{div}|_{H^{\text{div}}(\Omega)} \\ \text{grad}|_{H_{\Gamma_0}^1(\Omega)} & 0 \end{bmatrix}.$$

This shows that $\text{grad}|_{H_{\Gamma_0}^1(\Omega)}^* = -\left(\text{div}|_{H_{\Gamma_0}^{\text{div}}(\Omega)}\right)^* = -\text{div}|_{H_{\Gamma_0}^{\text{div}}(\Omega)}$, since $\text{div}|_{H_{\Gamma_0}^{\text{div}}(\Omega)}$ is closed. Looking at the upper right corners, we find $\text{grad}|_{H_0^1(\Omega)}^* = -\text{div}|_{H^{\text{div}}(\Omega)}$; this can also be obtained from the previous equality by taking $\Gamma_0 = \partial\Omega$. \square

The following theorem gives an example of how the general results in Section 2 can be applied to the wave equation.

Theorem 3.5. *For every accretive $\tilde{Q}_b \in \mathcal{L}(\mathcal{W}; \mathcal{W}')$, the operator $A_{\mathcal{H}}$ in (3.4) generates a contraction semigroup on $\mathcal{X}_{\mathcal{H}}$.*

Proof. We use Theorem 2.5, and we begin by identifying W_B . Since $\Psi_{\mathcal{W}}$ is injective,

$$\begin{aligned} \text{dom}(A)_{\mathcal{H}} &= \left\{ \begin{bmatrix} g \\ f \end{bmatrix} \in \mathcal{H}^{-1} \begin{bmatrix} H_{\Gamma_0}^1(\Omega) \\ H_{\text{div}}(\Omega) \end{bmatrix} \mid \Psi_{\mathcal{W}} \tilde{Q}_b \gamma_0 M_{1/\rho} g + \Psi_{\mathcal{W}} \gamma_{\perp} M_T f = 0 \right\} \\ &= \ker \left(\begin{bmatrix} \Psi_{\mathcal{W}} \tilde{Q}_b & I_{\mathcal{W}} \end{bmatrix} \begin{bmatrix} B_0 \\ \Psi_{\mathcal{W}} B_{\perp} \end{bmatrix} \right); \end{aligned}$$

hence $W_1 = \Psi_{\mathcal{W}} \tilde{Q}_b|_{\mathcal{W}}$ and $W_2 = I_{\mathcal{W}}$ with $\mathcal{K} = \mathcal{W}$. We next verify that these operators satisfy (2.6) and (2.11), starting with the latter. We have for all $k \in \mathcal{W}$ that

$$\langle (W_1 W_2^* + W_2 W_1^*) k, k \rangle_{\mathcal{W}} = 2 \text{Re} \langle \Psi_{\mathcal{W}} \tilde{Q}_b k, k \rangle_{\mathcal{W}} = 2 \text{Re} \langle \tilde{Q}_b k, k \rangle_{\mathcal{W}', \mathcal{W}} \geq 0. \quad (3.8)$$

Since the operator $\Psi_{\mathcal{W}} \tilde{Q}_b$ is defined everywhere, the calculation (3.8) shows that $\Psi_{\mathcal{W}} \tilde{Q}_b$ is maximal accretive on \mathcal{W} . This implies that $\Psi_{\mathcal{W}} \tilde{Q}_b + I = W_1 + W_2$ is invertible in \mathcal{W} ; hence (2.6) holds and $A_{\mathcal{H}}$ generates a contraction semigroup on $\mathcal{X}_{\mathcal{H}}$ by Theorem 2.5. \square

It is as usual straightforward to verify directly that $A_{\mathcal{H}}$ is dissipative if \tilde{Q}_b is dissipative, and so we might as well have used part (5) of Theorem 2.3 instead of Theorem 2.5. The preceding result should be compared to [TW09, §3.9].

4 The wave equation as a conservative boundary control system

In this section we show that, depending on the choices of the input and output spaces, we can interpret (1.1) as an impedance passive boundary control system in two different ways. First we briefly recall some central concepts on boundary control systems; see e.g. [MS07, §2] for more details.

Definition 4.1. A triple (L, K, G) of operators is an *(internally well-posed) boundary control system* on the triple $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$ of Hilbert spaces if it has the following properties:

1. The linear operators L , K and G have the same domain $\mathcal{Z} \subset \mathcal{X}$ and take values in \mathcal{X} , \mathcal{Y} and \mathcal{U} , respectively. The space \mathcal{Z} is endowed with the graph norm of $\begin{bmatrix} L \\ K \\ G \end{bmatrix}$ and it is called the *solution space*.
2. The operator $\begin{bmatrix} L \\ K \\ G \end{bmatrix}$ is closed from \mathcal{X} into $\begin{bmatrix} \mathcal{X} \\ \mathcal{Y} \\ \mathcal{U} \end{bmatrix}$.

3. The operator G is surjective.
4. The operator $A := L|_{\ker(G)}$ generates a strongly continuous semigroup on \mathcal{X} .

The boundary control system is *strong* if L is a closed operator on \mathcal{X} .

As was proved in [MS06, Lemma 2.6], a boundary control system (L, K, G) on $(\mathcal{U}, \mathcal{X}, \mathcal{Y})$ with solution space \mathcal{Z} has the following solvability property: For all initial states $z_0 \in \mathcal{Z}$ and input signals $u \in C^2(\mathbb{R}_+; \mathcal{U})$ compatible with z_0 , i.e., $u(0) = Gz_0$, the following system has a unique state trajectory $z \in C^1(\mathbb{R}_+; \mathcal{X}) \cap C(\mathbb{R}_+; \mathcal{Z})$ and the corresponding output signal satisfies $y \in C(\mathbb{R}_+; \mathcal{Y})$:

$$\dot{z}(t) = Lz(t), \quad u(t) = Gz(t), \quad \text{and} \quad y(t) = Kz(t), \quad t \geq 0, \quad z(0) = z_0.$$

Thus for those initial conditions and inputs, the above differential equation possesses a unique classical solution.

A boundary control system $\Xi := (L, K, G)$ is called *time-flow invertible* if the triple $\Xi^\leftarrow := (-L, G, K)$, the so called *time-flow inverse*, is also a boundary control system. The following definition is adapted from [MS07, §§2–3]:

Definition 4.2. A boundary control system is *scattering passive* if

$$2\operatorname{Re} \langle z, Lz \rangle_{\mathcal{X}} + \|Kz\|_{\mathcal{Y}}^2 \leq \|Gz\|_{\mathcal{U}}^2, \quad z \in \mathcal{Z}, \quad (4.1)$$

which holds if and only if all the classical solutions described above satisfy

$$\|x(T)\|^2 + \int_0^T \|y(t)\|^2 dt \leq \|x(0)\|^2 + \int_0^T \|u(t)\|^2 dt, \quad T \geq 0.$$

A boundary control system Ξ is called *scattering energy preserving* if we have equality in (4.1), and if in addition Ξ^\leftarrow is also a scattering-energy preserving boundary control system, then Ξ is called *scattering conservative*.

A boundary control system (L, K, G) with input space \mathcal{U} and output space \mathcal{Y} is *impedance passive* (*impedance conservative*) if there exists a unitary operator $\Psi : \mathcal{U} \rightarrow \mathcal{Y}$, such that the so-called *external Cayley transform* $(L, \tilde{K}, \tilde{G})$ is a scattering passive (scattering conservative) boundary control system, where

$$\tilde{K} := \frac{\Psi G - K}{\sqrt{2}} \quad \text{and} \quad \tilde{G} := \frac{\Psi G + K}{\sqrt{2}}. \quad (4.2)$$

The following two results formalise (1.1) as an impedance passive boundary control system in two different ways. First we assume that $\Gamma_1 = \emptyset$, i.e., that there is no damping on the boundary.

Corollary 4.3. *Let $\mathcal{Z} := \mathcal{H}^{-1} \begin{bmatrix} H_{\Gamma_0}^1(\Omega) \\ H^{\text{div}}(\Omega) \end{bmatrix}$, $L := \begin{bmatrix} 0 & \text{div} \\ \text{grad} & 0 \end{bmatrix} \mathcal{H}$, $K := B_0$, and $G := B_\perp$. Then (L, K, G) is an internally well-posed strong impedance conservative boundary control system with state space $\mathcal{X}_{\mathcal{H}}$, input space $\mathcal{U} = \mathcal{W}'$, and output space $\mathcal{Y} = \mathcal{W}$.*

Proof. The system (L, K, G) is an impedance conservative strong boundary control system by Theorem 3.2 and [MS07, Thm 5.2]. \square

In the case of Corollary 4.3, the operator Ψ in (4.2) converts forces into velocities; here we use $\Psi = \Psi_{\mathcal{W}}$ in (A.4). If we instead choose $L^2(\Gamma_2)$ as input and output space, then we can drop the assumption $\Gamma_1 = \emptyset$:

Theorem 4.4. *Assume that Γ_k , $k = 0, 1, 2$, form a splitting of $\partial\Omega$ with thin boundaries and let L , K , and G be as in Corollary 4.3. Let $Q_b \in \mathcal{L}(L^2(\Gamma_1))$ and define*

$$\mathcal{Z}_Q := \left\{ z \in \mathcal{H}^{-1} \begin{bmatrix} H_{\Gamma_0}^1(\Omega) \\ H^{\text{div}}(\Omega) \end{bmatrix} \mid Gz \in L^2(\Gamma_1 \cup \Gamma_2) \wedge Q_b \pi_1 Kz + \pi_1 Gz = 0 \right\} \quad (4.3)$$

with the graph norm of $\Xi_Q := \begin{bmatrix} L_Q \\ K_Q \\ G_Q \end{bmatrix} := \begin{bmatrix} L \\ \pi_2 K \\ \pi_2 G \end{bmatrix} \Big|_{\mathcal{Z}_Q}$.

Then (L_Q, K_Q, G_Q) is an internally well-posed impedance passive boundary control system with state space $\mathcal{X}_{\mathcal{H}}$ and input/output space $\mathcal{U}_Q = L^2(\Gamma_2)$. This system is impedance conservative if and only if Q_b is skew-adjoint. The system is strong if and only if $\mathcal{U}_Q = \{0\}$.

Proof. By Definition 4.2, it suffices to verify that the external Cayley transform $\left(L_Q, \frac{1}{\sqrt{2}}(G_Q - K_Q), \frac{1}{\sqrt{2}}(G_Q + K_Q) \right)$ of (L_Q, K_Q, G_Q) is a scattering passive (conservative) boundary control system. This can, according to [ALM13, Prop. 2.4], be achieved by establishing the inequality

$$2\text{Re} \langle L_Q z, z \rangle_{\mathcal{X}_{\mathcal{H}}} + \left\| \frac{G_Q - K_Q}{\sqrt{2}} z \right\|_{L^2(\Gamma_2)}^2 - \left\| \frac{G_Q + K_Q}{\sqrt{2}} z \right\|_{L^2(\Gamma_2)}^2 \leq 0, \quad z \in \mathcal{Z}_Q, \quad (4.4)$$

the surjectivity condition $(G_Q + K_Q)\mathcal{Z}_Q = L^2(\Gamma_2)$, and that $L_Q|_{\ker(G_Q + K_Q)}$ generates a contraction semigroup on \mathcal{X} . In order to prove conservativity, we additionally need to show that (4.4) holds with equality, that $(G_Q - K_Q)\mathcal{Z}_Q = \mathcal{X}$, and that $-L_Q|_{\ker(G_Q - K_Q)}$ generates a contraction semigroup on $\mathcal{X}_{\mathcal{H}}$. We do this in several steps.

Step 1 ($(G_Q \pm K_Q)\mathcal{Z}_Q = L^2(\Gamma_2)$) and (4.4) holds): Pick a $u \in L^2(\Gamma_2)$ arbitrarily and extend this u by zero on Γ_1 ; denote the result by \tilde{u} . Then $\tilde{u} \in L^2(\Gamma_1 \cup \Gamma_2)$, and by Theorem A.8 we can find an $f \in H^{\text{div}}(\Omega)$, such that $\gamma_\perp f = \tilde{u}$. Then $\mathcal{H}^{-1} \begin{bmatrix} 0 \\ f \end{bmatrix} \in \mathcal{Z}_Q$ and $(G_Q \pm K_Q)\mathcal{H}^{-1} \begin{bmatrix} 0 \\ f \end{bmatrix} = \pi_2 \gamma_\perp f = u$.

The left-hand side of (4.4) can for every $z \in \mathcal{Z}_Q$ be rewritten as

$$\begin{aligned}
2\operatorname{Re} \langle L_Q z, z \rangle_{\mathcal{X}} - 2\operatorname{Re} \langle G_Q z, K_Q z \rangle_{L^2(\Gamma_2)} &= 2\operatorname{Re} \langle Lz, z \rangle_{\mathcal{X}} - 2\operatorname{Re} \langle \pi_2 Gz, \pi_2 Kz \rangle_{L^2(\Gamma_2)} = \\
&= 2\operatorname{Re} \langle Lz, z \rangle_{\mathcal{X}} - 2\operatorname{Re} \langle B_{\perp} z, B_0 z \rangle_{\mathcal{W}', \mathcal{W}} + 2\operatorname{Re} \langle \pi_1 Gz, \pi_1 Kz \rangle_{L^2(\Gamma_1)} = \\
&= 2\operatorname{Re} \langle Lz, z \rangle_{\mathcal{X}} - 2\operatorname{Re} \langle \Psi_{\mathcal{W}} B_{\perp} z, B_0 z \rangle_{\mathcal{W}} + 2\operatorname{Re} \langle \pi_1 Gz, \pi_1 Kz \rangle_{L^2(\Gamma_1)} = \\
&= 2\operatorname{Re} \langle \pi_1 Gz, \pi_1 Kz \rangle_{L^2(\Gamma_1)} = -2\operatorname{Re} \langle Q_b \pi_1 Kz, \pi_1 Kz \rangle_{L^2(\Gamma_1)} \leq 0,
\end{aligned} \tag{4.5}$$

where we used (3.6), (4.3), and that Q_b is accretive on $L^2(\Gamma_1)$.

Step 2 ($L_Q|_{\ker(G_Q + K_Q)}$ generates a contraction semigroup): We use Theorem 2.6 and start by verifying that $L_Q|_{\ker(G_Q + K_Q)}$ is a closed operator on $\mathcal{X}_{\mathcal{H}}$. Let therefore $z_k \in \ker(G_Q + K_Q)$ tend to z in $\mathcal{X}_{\mathcal{H}}$, so that $\mathcal{H}z_k \rightarrow \mathcal{H}z$ in $L^2(\Omega)^{n+1}$. Let moreover $L_Q z_k = \begin{bmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{bmatrix} \mathcal{H}z_k \rightarrow v$ in $\mathcal{X}_{\mathcal{H}}$, hence in $L^2(\Omega)^{n+1}$. By the closedness of $\begin{bmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{bmatrix}$, we have $\mathcal{H}z_k \rightarrow \mathcal{H}z$ in $\begin{bmatrix} H^1(\Omega) \\ H^{\operatorname{div}}(\Omega) \end{bmatrix}$ and $v = \begin{bmatrix} 0 & \operatorname{div} \\ \operatorname{grad} & 0 \end{bmatrix} \mathcal{H}z$. Since $\begin{bmatrix} K \\ G \end{bmatrix} \mathcal{H}^{-1}$ is bounded from $\begin{bmatrix} H^1(\Omega) \\ H^{\operatorname{div}}(\Omega) \end{bmatrix}$ into $\begin{bmatrix} \mathcal{W} \\ \mathcal{W}' \end{bmatrix}$, we have $\begin{bmatrix} K \\ G \end{bmatrix} z_k \rightarrow \begin{bmatrix} K \\ G \end{bmatrix} z$ in $\begin{bmatrix} \mathcal{W} \\ \mathcal{W}' \end{bmatrix}$. On the other hand, because $z_k \in \ker(G_Q + K_Q)$,

$$Gz = \lim_{k \rightarrow \infty} (\pi_1 + \pi_2)Gz_k = \lim_{k \rightarrow \infty} -Q_b \pi_1 Kz_k - \pi_2 Kz_k = -(Q_b \pi_1 + \pi_2)Kz,$$

where the limits are taken in \mathcal{W}' . This shows that $Gz \in L^2(\Gamma_1 \cup \Gamma_2)$, $\pi_1 Gz + Q_b \pi_1 Kz = 0$, and $G_Q z + K_Q z = 0$. Thus, $z \in \ker(G_Q + K_Q)$ and $L_Q z = v$, i.e., L_Q is closed.

Furthermore, we have

$$\ker(G_Q + K_Q) = \ker \left(\begin{bmatrix} Q_b \pi_1 & \pi_1 \\ \pi_2 & \pi_2 \end{bmatrix} \begin{bmatrix} B_0 \\ B_{\perp} \end{bmatrix} \right) \tag{4.6}$$

and this space equals \mathcal{A} in (2.13) with $A_0^* = L$, $B_1 = K$, $b_2 = G$, $\mathcal{B}_0 = L^2(\Gamma_1 \cup \Gamma_2)$, $V_1 = \begin{bmatrix} Q_b \pi_1 \\ \pi_2 \end{bmatrix}$, $V_2 = \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix}$, and $\mathcal{K} = \begin{bmatrix} L^2(\Gamma_1) \\ L^2(\Gamma_2) \end{bmatrix}$. By Theorem 2.6, it is sufficient to show that $\ker \begin{bmatrix} V_1 & V_2 \end{bmatrix}$ is a dissipative relation in $L^2(\Gamma_1 \cup \Gamma_2)$ and that $V_1 V_2^* + V_2 V_1^* \geq 0$.

The following verifies that $\begin{bmatrix} u \\ v \end{bmatrix} \in \ker \begin{bmatrix} V_1 & V_2 \end{bmatrix} \implies \operatorname{Re} \langle u, v \rangle_{L^2(\Gamma_1 \cup \Gamma_2)} \leq 0$:

$$\begin{aligned}
\begin{bmatrix} Q_b \pi_1 & \pi_1 \\ \pi_2 & \pi_2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0 &\implies \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} v = - \begin{bmatrix} Q_b \pi_1 \\ \pi_2 \end{bmatrix} u \implies \\
\operatorname{Re} \left\langle \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} u, \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} v \right\rangle &= -\operatorname{Re} \left\langle \begin{bmatrix} \pi_1 u \\ \pi_2 u \end{bmatrix}, \begin{bmatrix} Q_b \pi_1 u \\ \pi_2 u \end{bmatrix} \right\rangle \leq 0.
\end{aligned}$$

Moreover, $V_2^* = \begin{bmatrix} \mathcal{I}_1 & \mathcal{I}_2 \end{bmatrix} : \mathcal{K} \rightarrow L^2(\Gamma_1 \cup \Gamma_2)$, where \mathcal{I}_k is the appropriate injection, and hence for all $r \in L^2(\Gamma_1)$, $s \in L^2(\Gamma_2)$:

$$\operatorname{Re} \left\langle V_1 V_2^* \begin{bmatrix} r \\ s \end{bmatrix}, \begin{bmatrix} r \\ s \end{bmatrix} \right\rangle = \operatorname{Re} \left\langle \begin{bmatrix} Q_b \pi_1(r+s) \\ \pi_2(r+s) \end{bmatrix}, \begin{bmatrix} r \\ s \end{bmatrix} \right\rangle = \operatorname{Re} \left\langle \begin{bmatrix} Q_b r \\ s \end{bmatrix}, \begin{bmatrix} r \\ s \end{bmatrix} \right\rangle \geq 0.$$

Theorem 2.6 now completes step 2.

Step 3 (Ξ_Q is impedance conservative iff $Q_b^ = -Q_b$):* First assume that Ξ_Q is impedance conservative; then (4.5) holds with equality. If we can establish that $KZ_Q = \mathcal{W}$, then $\pi_1 KZ_Q$ is dense in $L^2(\Gamma_1)$, and it follows from the last equality in (4.5) and the boundedness of Q_b that $Q_b^* = -Q_b$. We pick $w \in \mathcal{W}$ arbitrarily and choose $g \in M_\rho H_{\Gamma_0}^1(\Omega)$ such that $\gamma_0 M_{1/\rho} g = w$. Further choosing $f \in M_T^{-1} H^{\text{div}}(\Omega)$, such that $\gamma_\perp M_T f = Q_b \pi_1 w$, we obtain $\begin{bmatrix} g \\ f \end{bmatrix} \in Z_Q$ and $K \begin{bmatrix} g \\ f \end{bmatrix} = \gamma_0 M_{1/\rho} g = w$.

Now conversely assume that $Q_b^* = -Q_b$. Then we have equality in (4.5) and the argument in step 2 (with a few changes of signs) shows that $-L_Q|_{\ker(G_Q - K_Q)}$ generates a contraction semigroup on $\mathcal{X}_\mathcal{H}$.

Step 4 (The remaining claims): Internal well-posedness, i.e., that $L_Q|_{\ker(G_Q)}$ generates a contraction semigroup, is proved using exactly the same argument as in Step 2, but with $V_1 = \begin{bmatrix} Q_b \pi_1 \\ 0 \end{bmatrix}$, since

$$\ker(G_Q) = \ker \left(\begin{bmatrix} Q_b \pi_1 & \pi_1 \\ 0 & \pi_2 \end{bmatrix} \right).$$

It remains to show that Ξ_Q is strong if and only if $\mathcal{U}_Q = \{0\}$. If $\mathcal{U}_Q = \{0\}$, then $\begin{bmatrix} L_Q \\ 0 \\ 0 \end{bmatrix} = \Xi_Q$, which is a boundary control system by the above, hence closed by Definition 4.1; then L_Q is a closed operator. If $\mathcal{U}_Q \neq \{0\}$, then we can choose a $\mu \in \overline{L^2(\Gamma_2)} \setminus L^2(\Gamma_2)$, where the closure is taken in \mathcal{W}' , and pick a sequence $\mu_k \in L^2(\Gamma_2)$ such that $\mu_k \rightarrow \mu$ in \mathcal{W}' . Next we define a sequence in Z_Q that converges in the graph norm of L_Q by setting $\begin{bmatrix} g_k \\ f_k \end{bmatrix} := \mathcal{H}^{-1} \begin{bmatrix} 0 \\ R\mu_k \end{bmatrix}$, where γ_\perp^{-r} is any bounded right inverse of γ_\perp . The limit $\mathcal{H}^{-1} \begin{bmatrix} 0 \\ \gamma_\perp^{-r} \mu \end{bmatrix}$ of this sequence has the property $G\mathcal{H}^{-1} \begin{bmatrix} 0 \\ \gamma_\perp^{-r} \mu \end{bmatrix} = \mu \notin L^2(\Gamma_1 \cup \Gamma_2)$, hence $\begin{bmatrix} 0 \\ \gamma_\perp^{-r} \mu \end{bmatrix} \notin Z_Q$ and so L_Q is not closed. \square

The following result connects the classical solutions of (1.1) to those of (L_Q, K_Q, G_Q) :

Theorem 4.5. *Let $u \in C^2(\mathbb{R}_+; L^2(\Gamma_2))$ and $z_0, w_0 \in L^2(\Omega)$ be such that $\begin{bmatrix} M_\rho w_0 \\ \text{grad } z_0 \end{bmatrix} \in Z_Q$ and $G_Q \begin{bmatrix} M_\rho w_0 \\ \text{grad } z_0 \end{bmatrix} = u(0)$. Then the unique classical solution $\begin{bmatrix} g \\ f \end{bmatrix}$ of*

$$\frac{d}{dt} \begin{bmatrix} g(t) \\ f(t) \end{bmatrix} = L_Q \begin{bmatrix} g(t) \\ f(t) \end{bmatrix}, \quad G_Q \begin{bmatrix} g(t) \\ f(t) \end{bmatrix} = u(t), \quad t \geq 0, \quad \begin{bmatrix} g(0) \\ f(0) \end{bmatrix} = \begin{bmatrix} M_\rho w_0 \\ \text{grad } z_0 \end{bmatrix}, \quad (4.7)$$

$$\begin{aligned} \text{satisfies } M_{1/\rho} g &\in C(\mathbb{R}_+; H_{\Gamma_0}^1(\Omega)) \cap C^1(\mathbb{R}_+; L^2(\Omega)) \quad \text{and} \\ M_T f &\in C(\mathbb{R}_+; H^{\text{div}}(\Omega)) \cap C^1(\mathbb{R}_+; L^2(\Omega)^n); \end{aligned} \quad (4.8)$$

in particular $\text{grad}(M_{1/\rho}g) \in C(\mathbb{R}_+; L^2(\Omega)^n)$. Defining

$$y(t) := K_Q \begin{bmatrix} g(t) \\ f(t) \end{bmatrix} \quad \text{and} \quad z(\xi, t) := z_0(\xi) + \int_0^t \frac{g(\xi, s)}{\rho(\xi)} ds, \quad \xi \in \Omega, \quad t \geq 0,$$

we obtain that $y \in C(\mathbb{R}_+; L^2(\Gamma_2))$,

$$\begin{aligned} z &\in C^1(\mathbb{R}_+; H_{\Gamma_0}^1(\Omega)) \cap C^2(\mathbb{R}_+; L^2(\Omega)) \quad \text{and} \\ \text{div}(M_T \text{grad} z(\cdot)) &\in C(\mathbb{R}_+; L^2(\Omega)), \end{aligned} \tag{4.9}$$

and that (z, y) solves (1.1) with $Q_i = 0$. Note that by Proposition A.4 we interpret the boundary mapping, $\nu \cdot (T(\cdot) \text{grad} z(\cdot, t))$, as $\gamma_\perp(M_T \text{grad} z(t))$.

Proof. By the standard smoothness property of the state trajectory of a boundary control system mentioned after Definition 4.1, $\begin{bmatrix} g \\ f \end{bmatrix} \in C^1(\mathbb{R}_+; \mathcal{X}_H) \cap C(\mathbb{R}_+; \mathcal{Z}_Q)$, and in particular $\mathcal{H} \begin{bmatrix} g \\ f \end{bmatrix} \in C\left(\mathbb{R}_+; \begin{bmatrix} H_{\Gamma_0}^1(\Omega) \\ H^{\text{div}}(\Omega) \end{bmatrix}\right)$, i.e., (4.8) holds. We have

$$y = K_Q \begin{bmatrix} g \\ f \end{bmatrix} = \pi_2 \gamma_0 M_{1/\rho} g$$

which is in $C(\mathbb{R}_+; L^2(\Gamma_2))$ by (4.8) and the continuity of $\pi_2 \gamma_0 : H^1(\Omega) \rightarrow L^2(\Gamma_2)$.

From the definition of z , it follows immediately that $\dot{z}(t) = M_{1/\rho} g(t)$ and that (using (4.7))

$$\text{grad} z(t) = \text{grad} z_0 + \int_0^t \text{grad}(M_{1/\rho} g(s)) ds = \text{grad} z_0 + \int_0^t \dot{f}(s) ds = f(t).$$

This implies that

$$M_\rho \ddot{z}(t) = \dot{g}(t) = \text{div}(M_T f(t)) = \text{div}(M_T \text{grad} z(t)),$$

and so $\text{div}(M_T \text{grad} z(\cdot)) \in C(\mathbb{R}_+; L^2(\Omega))$ since $z \in C^2(\mathbb{R}_+; L^2(\Omega))$, which proves (4.9). Moreover, (y, z) solves (1.1) with $Q_i = 0$. \square

Remark 4.6. By Definition 4.2 and Theorem 4.4, the system

$$\left\{ \begin{aligned} \rho(\xi) \frac{\partial^2 z}{\partial t^2}(\xi, t) &= \text{div}(T(\xi) \text{grad} z(\xi, t)) - \left(Q_i \frac{\partial z}{\partial t}\right)(\xi, t), \quad \xi \in \Omega, \quad t \geq 0, \\ 0 &= \frac{\partial z}{\partial t}(\xi, t) \quad \text{on } \Gamma_0 \times \mathbb{R}_+, \\ 0 &= \nu \cdot (T(\xi) \text{grad} z(\xi, t)) + \left(Q_b \frac{\partial z}{\partial t}\right)(\xi, t) \quad \text{on } \Gamma_1 \times \mathbb{R}_+, \\ \sqrt{2}u(\xi, t) &= \nu \cdot (T(\xi) \text{grad} z(\xi, t)) + \frac{\partial z}{\partial t}(\xi, t) \quad \text{on } \Gamma_2 \times \mathbb{R}_+, \\ \sqrt{2}y(\xi, t) &= \nu \cdot (T(\xi) \text{grad} z(\xi, t)) - \frac{\partial z}{\partial t}(\xi, t) \quad \text{on } \Gamma_2 \times \mathbb{R}_+, \\ z(\xi, 0) &= z_0(\xi), \quad \frac{\partial z}{\partial t}(\xi, 0) = w_0(\xi) \quad \text{on } \Omega, \end{aligned} \right. \tag{4.10}$$

is a scattering-passive boundary control system with state $\begin{bmatrix} \rho(\cdot)\dot{z} \\ \text{grad } z \end{bmatrix}$, input u , and output y , and in particular it is L^2 -well-posed. The state space is $\mathcal{X}_{\mathcal{H}}$ and the input/output space is $L^2(\Gamma_2)$. The system (4.10) is even scattering conservative if $Q_i = 0$ and $Q_b^* = -Q_b$. The statements in Theorem 4.5 remain true for the scattering representation if one replaces all occurrences of G_Q and K_Q by $\frac{1}{\sqrt{2}}(G_Q + K_Q)$ and $\frac{1}{\sqrt{2}}(G_Q - K_Q)$, respectively. The pair (z, y) then solves (4.10) with $Q_i = 0$ instead of (1.1).

The scattering-passive system (4.10) fits into the abstract framework developed for Maxwell's equations in [?, ?], at least in the case $\Gamma_1 = \emptyset$, i.e., when there is no damping at the boundary. In a forthcoming paper, we shall give more details on this.

References

- [AF03] Robert A. Adams and John J. F. Fournier, *Sobolev spaces*, second ed., Pure and Applied Mathematics (Amsterdam), vol. 140, Elsevier/Academic Press, Amsterdam, 2003.
- [ALM13] Atte Aalto, Teemu Lukkari, and Jarmo Malinen, *Acoustic wave guides as infinite-dimensional dynamical systems*, to appear in ESAIM:COCV, <http://arxiv.org/abs/1211.7000v2>, 2014.
- [Arl12] Yury Arlinskiĭ, *Boundary triplets and maximal accretive extensions of sectorial operators*, Operator methods for boundary value problems, London Math. Soc. Lecture Note Ser., vol. 404, Cambridge Univ. Press, Cambridge, 2012, pp. 35–72.
- [DHMdS09] V. A. Derkach, S. Hassi, M. M. Malamud, and H. de Snoo, *Boundary relations and generalized resolvents of symmetric operators*, Russ. J. Math. Phys. **16** (2009), no. 1, 17–60.
- [GG91] V. I. Gorbachuk and M. L. Gorbachuk, *Boundary value problems for operator differential equations*, Mathematics and its Applications (Soviet Series), vol. 48, Kluwer Academic Publishers Group, Dordrecht, 1991, translation and revised from the 1984 Russian original.
- [GR86] Vivette Girault and Pierre-Arnaud Raviart, *Finite element methods for Navier-Stokes equations: Theory and algorithms*, Springer Series in Computational Mathematics, vol. 5, Springer-Verlag, Berlin, 1986.
- [Gri85] Pierre Grisvard, *Elliptic problems in nonsmooth domains*, Monographs and Studies in Mathematics, vol. 24, Pitman (Advanced Publishing Program), Boston, MA, 1985.

- [GZM05] Yann Le Gorrec, Hans Zwart, and Bernhard Maschke, *Dirac structures and boundary control systems associated with skew-symmetric differential operators*, SIAM J. Control Optim. **44** (2005), no. 5, 1864–1892 (electronic).
- [JZ12] Birgit Jacob and Hans Zwart, *Linear port-hamiltonian systems on infinite-dimensional spaces*, Operator Theory: Advances and Applications, vol. 223, Birkhäuser-Verlag, 2012.
- [KZ12] Mikael Kurula and Hans Zwart, *The duality between the gradient and divergence operators on bounded Lipschitz domains*, Memorandum 1994, Department of Applied Mathematics, University of Twente, Enschede, October 2012.
- [LM72] J.-L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applications. Vol. I*, Springer-Verlag, New York, 1972, translation from the French by P. Kenneth, Die Grundlehren der mathematischen Wissenschaften, Band 181.
- [MS06] Jarmo Malinen and Olof J. Staffans, *Conservative boundary control systems*, J. Differential Equations **231** (2006), 290–312.
- [MS07] ———, *Impedance passive and conservative boundary control systems*, Complex Anal. Oper. Theory **1** (2007), 279–30.
- [Paz83] Amnon Pazy, *Semi-groups of linear operators and applications to partial differential equations*, Springer-Verlag, Berlin, 1983.
- [Neč12] Jindřich Nečas, *Direct methods in the theory of elliptic partial differential equations*, Springer Verlag, Heidelberg Berlin, 2012, Corrected 2nd printing.
- [RR93] Michael Renardy and Robert C. Rogers, *An introduction to partial differential equations*, Texts in Applied Mathematics, vol. 13, Springer-Verlag, New York, 1993.
- [Spi65] Michael Spivak, *Calculus on manifolds. A modern approach to classical theorems of advanced calculus*, W. A. Benjamin, Inc., New York-Amsterdam, 1965.
- [Tro13] Sascha Trostorff, *Autonomous evolutionary inclusions with applications to problems with nonlinear boundary conditions*, Int. J. Pure Appl. Math. **85** (2013), no. 2, 303–338.
- [Tro14] ———, *A characterization of boundary conditions yielding maximal monotone operators*, posted in arXiv, <http://arxiv.org/abs/1310.5014>, 2014.

- [TW09] Marius Tucsnak and George Weiss, *Observation and control for operator semigroups*, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser Verlag, Basel, 2009, (electronic version).
- [Yos95] Kōsaku Yosida, *Functional analysis*, Classics in Mathematics, Springer-Verlag, Berlin, 1995, Reprint of the sixth (1980) edition.

A Sobolev-space background

The necessary background for the present article has been compiled in [KZ12]. Here we only fix the notation briefly and the reader is referred to [KZ12] for more details. We mainly cite [TW09] for convenience; references to the standard sources, such as [Spi65, Neč12, AF03, Gri85, LM72], can be found there. To the best of our knowledge, Proposition A.6 and Theorems A.7–A.8 are new for the case $\mathcal{W}' \neq H^{-1/2}(\partial\Omega)$, i.e., for Γ_0 with positive surface measure.

A *bounded Lipschitz set* is a bounded and open subset Ω of \mathbb{R}^n which has a Lipschitz-continuous boundary; see [TW09, §13]. By $\mathcal{D}(\Omega)$ we mean the space of test functions on Ω , i.e., functions in $C^\infty(\Omega)$ with compact support contained in Ω , and $\mathcal{D}'(\Omega)$ denotes the set of distributions on Ω .

Definition A.1. The *divergence operator* is the operator $\operatorname{div} : \mathcal{D}'(\Omega)^n \rightarrow \mathcal{D}'(\Omega)$ given by

$$\operatorname{div} v = \frac{\partial v_1}{\partial x_1} + \dots + \frac{\partial v_n}{\partial x_n},$$

and the *gradient operator* is the operator $\operatorname{grad} : \mathcal{D}'(\Omega) \rightarrow \mathcal{D}'(\Omega)^n$ defined by

$$\operatorname{grad} w = \left(\frac{\partial w}{\partial x_1}, \dots, \frac{\partial w}{\partial x_n} \right)^\top.$$

The *Laplacian* is defined as a linear operator on $\mathcal{D}'(\Omega)$ by $\Delta x := \operatorname{div}(\operatorname{grad} x)$.

The Sobolev space $H^1(\Omega)$ is as usual defined as the space

$$H^1(\Omega) := \{v \in L^2(\Omega) \mid \operatorname{grad} v \in L^2(\Omega)^n\}$$

equipped with the graph norm of grad . Similarly, we define

$$H^{\operatorname{div}}(\Omega) := \{v \in L^2(\Omega)^n \mid \operatorname{div} v \in L^2(\Omega)\},$$

equipped with the graph norm of div . These are the maximal domains for which grad and div can be considered as operators between L^2 spaces.

Definition A.2. The closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$ is denoted by $H_0^1(\Omega)$ and the closure of $\mathcal{D}(\Omega)^n$ in $H^{\operatorname{div}}(\Omega)$ is denoted by $H_0^{\operatorname{div}}(\Omega)$.

It is easy to see that $H^1(\Omega)^n \subset H^{\text{div}}(\Omega) \subset L^2(\Omega)^n$ with continuous embeddings. It is well known that $\mathcal{D}(\overline{\Omega})^n$, the restrictions to the closure of Ω of all functions in $C^\infty(\mathbb{R}^n)$, is dense in $L^2(\Omega)$; see e.g. [GR86, Thm I.1.2.1]. Hence, $H^{\text{div}}(\Omega)$ is dense in $L^2(\Omega)^n$, and due to the following lemma the other embedding is also dense:

Lemma A.3. *Let Ω be a subset of \mathbb{R}^n with Lipschitz-continuous boundary. Then $\mathcal{D}(\overline{\Omega})^n$ is dense in $H^{\text{div}}(\Omega)$. It follows that also $H^1(\Omega)^n$ is dense in $H^{\text{div}}(\Omega)$.*

For proof, see [GR86, Thm I.2.4]. If Ω is a bounded Lipschitz set in \mathbb{R}^n , then the outward unit normal vector field is defined for almost all $x \in \partial\Omega$ using local coordinates, and we can define a vector field ν in a neighbourhood of $\overline{\Omega}$ that coincides with the outward unit normal vector field for almost every $x \in \partial\Omega$; see [TW09, Def. 13.6.3] and the remarks following. According to [TW09, pp. 424–425], we have $\nu \in L^\infty(\partial\Omega)^n$.

The space $H^{1/2}(\partial\Omega)$ is the Hilbert space of all functions in $L^2(\partial\Omega)$ with finite $H^{1/2}(\partial\Omega)$ norm, which is given by

$$\|f\|_{H^{1/2}(\partial\Omega)}^2 = \|f\|_{L^2(\partial\Omega)}^2 + \int_{\partial\Omega} \int_{\partial\Omega} \frac{|f(x) - f(y)|^2}{\|x - y\|_{\mathbb{R}^n}^n} d\sigma_x d\sigma_y, \quad (\text{A.1})$$

where $d\sigma$ is the surface measure on $\partial\Omega$; see [KZ12, §4] or [TW09, pp. 422–423] for more details. The space $H^{-1/2}(\partial\Omega)$ is the dual of $H^{1/2}(\partial\Omega)$ with pivot space $L^2(\partial\Omega)$; see e.g. [TW09, §2.9].

The following result is a consequence of [GR86, Thm I.1.5]:

Lemma A.4. *For a bounded Lipschitz set Ω , the boundary trace mapping $g \mapsto g|_{\partial\Omega} : \mathcal{D}(\overline{\Omega}) \rightarrow C(\partial\Omega)$ has a unique continuous extension γ_0 that maps $H^1(\Omega)$ onto $H^{1/2}(\partial\Omega)$. The space $H_0^1(\Omega)$ in Definition A.2 equals $\{g \in H^1(\Omega) \mid \gamma_0 g = 0\}$.*

We call γ_0 the *Dirichlet trace map*. In the following integration by parts formula, the dot \cdot denotes the inner product in \mathbb{R}^n , $p \cdot q = q^\top p$ without complex conjugate:

Lemma A.5. *Let Ω be a bounded Lipschitz subset of \mathbb{R}^n . Then*

$$\langle \text{div } f, g \rangle_{L^2(\Omega)} + \langle f, \text{grad } g \rangle_{L^2(\Omega)^n} = \int_{\partial\Omega} (\nu \cdot \gamma_0 f) \gamma_0 \overline{g} d\sigma. \quad (\text{A.2})$$

holds for arbitrary $f \in H^1(\Omega)^n$ and $g \in H^1(\Omega)$.

For a proof, see [TW09, Rem. 13.7.2]. Note that $f \in H^1(\Omega)^n$ implies that the boundary trace of f , $\gamma_0 f \in H^{1/2}(\partial\Omega)^n \subset L^2(\partial\Omega)^n$. Moreover, by the above it holds that $\nu \in L^\infty(\partial\Omega)^n$, and hence we obtain that $\nu \cdot \gamma_0 f \in L^2(\partial\Omega)$ for all $f \in H^1(\Omega)^n$.

In the sequel we make the standing assumption that Γ_0, Γ_\bullet forms a splitting of $\partial\Omega$ with thin boundaries; see Definition 3.1. The result statements remain true if Γ_\bullet is further split into subsets with thin boundaries, as we do in §§3–4. Following [TW09, §13.6], we write

$$H_{\Gamma_0}^1(\Omega) := \{g \in H^1(\Omega) \mid (\gamma_0 g)|_{\Gamma_0} = 0 \text{ in } L^2(\Gamma_0)\}. \quad (\text{A.3})$$

We can also write $H_{\Gamma_0}^1(\Omega) = \ker(\pi_0 \gamma_0)$, where π_0 is the orthogonal projection of $L^2(\partial\Omega)$ onto $L^2(\Gamma_0)$. Since $H^{1/2}(\partial\Omega)$ is continuously embedded in $L^2(\partial\Omega)$ by (A.1), the operator $\pi_0 \gamma_0 : H^1(\Omega) \rightarrow L^2(\Gamma_0)$ is bounded; hence $H_{\Gamma_0}^1(\Omega)$ is closed in $H^1(\Omega)$.

Obviously γ_0 maps $H_{\Gamma_0}^1(\Omega)$ onto $\mathcal{W} := \gamma_0 H_{\Gamma_0}^1(\Omega)$ with inner product inherited from $H^{1/2}(\partial\Omega)$. This space is dense in $L^2(\Gamma_\bullet)$ by [TW09, Thm 13.6.10 and Rem. 13.6.12], and it is immediate that the inclusion map is continuous. Denote the dual of \mathcal{W} with pivot space $L^2(\Gamma_\bullet)$ by \mathcal{W}' .

By the Riesz representation theorem, there exists a unitary operator $\Psi_{\mathcal{W}} : \mathcal{W}' \rightarrow \mathcal{W}$, such that

$$(x, z)_{\mathcal{W}', \mathcal{W}} = \langle \Psi_{\mathcal{W}} x, z \rangle_{\mathcal{W}} = \langle x, \Psi_{\mathcal{W}}^* z \rangle_{\mathcal{W}'} \quad (\text{A.4})$$

for all $x \in \mathcal{W}'$ and $z \in \mathcal{W}$; see [TW09, p. 57] and [MS07, p. 288–289]. Thus \mathcal{W}' is also a Hilbert space, with inner product

$$\langle u, v \rangle_{\mathcal{W}'} = \langle \Psi_{\mathcal{W}} u, \Psi_{\mathcal{W}} v \rangle_{\mathcal{W}}, \quad u, v \in \langle u, v \rangle_{\mathcal{W}'}.$$

The operator $\Psi_{\mathcal{W}}$ can alternatively be characterised as the operator in $\mathcal{L}(\mathcal{W}'; \mathcal{W})$ uniquely determined by

$$\langle \Psi_{\mathcal{W}} x, z \rangle_{\mathcal{W}} = \lim_{n \rightarrow \infty} \langle x_n, z \rangle_{L^2(\Gamma_\bullet)}, \quad x \in \mathcal{W}', \quad z \in \mathcal{W},$$

where $x_n \in L^2(\Gamma_\bullet)$ is an arbitrary sequence converging to x in \mathcal{W}' ; see [TW09, §2.9].

Proposition A.6. *For a bounded Lipschitz set Ω , the restricted normal trace map $u \mapsto (\nu \cdot u)|_{\Gamma_\bullet} : \mathcal{D}(\overline{\Omega})^n \rightarrow L^2(\Gamma_\bullet)$ has a unique continuous extension γ_\perp that maps $H^{\text{div}}(\Omega)$ into \mathcal{W}' .*

Proof. We follow the argument in [GR86, Thm I.2.5] with some small modifications. By (A.2), we have

$$\begin{aligned} \left| \int_{\partial\Omega} (\nu \cdot u) \overline{\phi} \, d\sigma \right| &\leq \left| \langle \text{div } u, \phi \rangle_{L^2(\Omega)} \right| + \left| \langle u, \text{grad } \phi \rangle_{L^2(\Omega)^n} \right| \\ &\leq \|\text{div } u\|_{L^2(\Omega)} \|\phi\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)^n} \|\text{grad } \phi\|_{L^2(\Omega)^n} \\ &\leq 2\|u\|_{H^{\text{div}}(\Omega)} \|\phi\|_{H^1(\Omega)}, \quad u \in \mathcal{D}(\overline{\Omega})^n, \quad \phi \in H^1(\Omega). \end{aligned}$$

Denote an arbitrary continuous right inverse of γ_0 by γ_0^{-r} , choose an arbitrary $\mu \in \mathcal{W}$, and set $\phi := \gamma_0^{-r} \mu$. Since μ vanishes on Γ_0 , we obtain

$$\left| \int_{\Gamma_\bullet} (\nu \cdot u) \bar{\mu} \, d\sigma \right| = \left| \int_{\partial\Omega} (\nu \cdot u) \bar{\gamma}_0^{-r} \mu \, d\sigma \right| \leq 2 \|u\|_{H^{\text{div}}(\Omega)} \|\gamma_0^{-r}\| \|\mu\|_{\mathcal{W}},$$

i.e., that the restricted normal trace has operator norm at most $2\|\gamma_0^{-r}\|$ from $H^{\text{div}}(\Omega)$ into \mathcal{W}' . This restricted normal trace is defined densely in $H^{\text{div}}(\Omega)$ by Lemma A.3, and hence it can be extended uniquely to a bounded operator γ_\perp from $H^{\text{div}}(\Omega)$ into \mathcal{W}' . \square

The operator γ_\perp is referred to as the *(restricted) normal trace map*.

Theorem A.7. *Let Ω be a bounded Lipschitz set in \mathbb{R}^n . For all $f \in H^{\text{div}}(\Omega)$ and $g \in H_{\Gamma_0}^1(\Omega)$ it holds that*

$$\langle \text{div } f, g \rangle_{L^2(\Omega)} + \langle f, \text{grad } g \rangle_{L^2(\Omega)^n} = (\gamma_\perp f, \gamma_0 g)_{\mathcal{W}', \mathcal{W}}. \quad (\text{A.5})$$

In particular, we have the following Green's formula:

$$\langle \Delta h, g \rangle_{L^2(\Omega)} + \langle \text{grad } h, \text{grad } g \rangle_{L^2(\Omega)^n} = (\gamma_\perp \text{grad } h, \gamma_0 g)_{\mathcal{W}', \mathcal{W}}, \quad (\text{A.6})$$

which is valid for all $h \in H^1(\Omega)$ such that $\Delta h \in L^2(\Omega)$ and all $g \in H_{\Gamma_0}^1(\Omega)$.

Proof. Since $\pi_0 \gamma_0 g = 0$ for $g \in H_{\Gamma_0}^1(\Omega)$, we obtain from (A.2) that

$$\langle \text{div } f, g \rangle_{L^2(\Omega)} + \langle f, \text{grad } g \rangle_{L^2(\Omega)^n} = \langle \gamma_\perp f, \gamma_0 g \rangle_{L^2(\Gamma_\bullet)}$$

for $f \in H^1(\Omega)^n$ and $g \in H_{\Gamma_0}^1(\Omega)$. Using the fact that \mathcal{W}' is the dual of \mathcal{W} with pivot space $L^2(\Gamma_\bullet)$, we obtain (A.5) for $f \in H^1(\Omega)^n$ and $g \in H_{\Gamma_0}^1(\Omega)$.

For every $g \in H_{\Gamma_0}^1(\Omega)$, the mapping $u \mapsto (u, \gamma_0 g)_{\mathcal{W}', \mathcal{W}}$ is a bounded linear functional on \mathcal{W}' , and by Proposition A.6, γ_\perp maps $H^{\text{div}}(\Omega)$ continuously into \mathcal{W}' . Hence, if $f_n \in H^1(\Omega)^n$ tends to f in $H^{\text{div}}(\Omega)$, then $\text{div } f_n \rightarrow \text{div } f$ in $L^2(\Omega)$, $f_n \rightarrow f$ in $L^2(\Omega)^n$, and $\gamma_\perp f_n \rightarrow \gamma_\perp f$ in \mathcal{W}' . We can thus conclude that (A.5) holds for all $g \in H_{\Gamma_0}^1(\Omega)$ and all f in the closure of $H^1(\Omega)^n$ in $H^{\text{div}}(\Omega)$, i.e., for all $f \in H^{\text{div}}(\Omega)$; see Lemma A.3.

In order to prove (A.6), we let $h \in H^1(\Omega)$ be such that $\Delta h \in L^2(\Omega)$ and set $f := \text{grad } h$. Then $f \in L^2(\Omega)^n$ and $\text{div } (\text{grad } h) = \Delta h \in L^2(\Omega)$, so $f \in H^{\text{div}}(\Omega)$. Now (A.6) follows from (A.5). \square

If we take $\Gamma_0 = \emptyset$ in the preceding theorem, then we obtain a well-known special case. The next result gives the surjectivity of the normal trace map, and this critical for associating a boundary triplet to the wave equation.

Theorem A.8. *For a bounded Lipschitz set Ω , γ_\perp maps $H^{\text{div}}(\Omega)$ boundedly onto \mathcal{W}' .*

Proof. By Proposition A.6, γ_\perp maps $H^{\text{div}}(\Omega)$ boundedly into \mathcal{W}' , and it only remains to establish surjectivity. For this we use an adaptation of the proof of [GR86, Cor. I.2.]. First we fix an arbitrary $\mu \in \mathcal{W}'$ and using the Lax-Milgram theorem [Gri85, Lemma 2.2.1.1], we find a unique $\phi \in H_{\Gamma_0}^1(\Omega)$ which solves the following problem:

$$-\Delta\phi + \phi = 0 \quad \text{in } L^2(\Omega) \quad \text{and} \quad \gamma_\perp \text{grad } \phi = \mu. \quad (\text{A.7})$$

Indeed, the sesqui-linear form $(v, \phi) \mapsto \langle v, \phi \rangle_{H_{\Gamma_0}^1(\Omega)}$ is bounded and coercive on $H_{\Gamma_0}^1(\Omega)^2$, and the linear form $v \mapsto (\gamma_0 v, \mu)_{\mathcal{W}, \mathcal{W}'}$ is bounded on $H_{\Gamma_0}^1(\Omega)$ according to Lemma A.4. By the Lax-Milgram theorem there exists a unique $\phi \in H_{\Gamma_0}^1(\Omega)$, such that

$$\langle v, \phi \rangle_{H_{\Gamma_0}^1(\Omega)} = (\gamma_0 v, \mu)_{\mathcal{W}, \mathcal{W}'}, \quad v \in H_{\Gamma_0}^1(\Omega). \quad (\text{A.8})$$

Taking $v \in \mathcal{D}(\Omega)$, we by Lemma A.4 and Green's identity (A.6) obtain that for all $v \in \mathcal{D}(\Omega)$:

$$\begin{aligned} 0 &= \langle v, \phi \rangle_{H_{\Gamma_0}^1(\Omega)} = \langle v, \phi \rangle_{L^2(\Omega)} + \langle \text{grad } v, \text{grad } \phi \rangle_{L^2(\Omega)^n} \\ &= (v, \overline{\phi})_{\mathcal{D}(\Omega), \mathcal{D}(\Omega)'} + (\text{grad } v, \overline{\text{grad } \phi})_{\mathcal{D}(\Omega)^n, (\mathcal{D}(\Omega)')^n} = (v, \overline{(I - \Delta)\phi})_{\mathcal{D}(\Omega), \mathcal{D}(\Omega)'}, \end{aligned}$$

i.e., that $\Delta\phi = \phi$ in the sense of distributions on Ω , and hence in particular $\phi \in H_{\Gamma_0}^1(\Omega)$ with $\Delta\phi \in L^2(\Omega)$. Using this and (A.6) on (A.8), we thus obtain

$$\begin{aligned} (\gamma_0 v, \mu)_{\mathcal{W}, \mathcal{W}'} &= \langle v, \phi \rangle_{H_{\Gamma_0}^1(\Omega)} = \langle v, \phi \rangle_{L^2(\Omega)} + \langle \text{grad } v, \text{grad } \phi \rangle_{L^2(\Omega)^n} \\ &= \langle v, (I - \Delta)\phi \rangle_{L^2(\Omega)} + (\gamma_0 v, \gamma_\perp \text{grad } \phi)_{\mathcal{W}, \mathcal{W}'} \\ &= (\gamma_0 v, \gamma_\perp \text{grad } \phi)_{\mathcal{W}, \mathcal{W}'}, \quad v \in H_{\Gamma_0}^1(\Omega). \end{aligned} \quad (\text{A.9})$$

This proves that ϕ solves the problem (A.7). Now we set $v := \text{grad } \phi$, which lies in $H^{\text{div}}(\Omega)$, because $\text{div}(\text{grad } \phi) = \phi$ by (A.7). Furthermore, $\gamma_\perp v = \mu$ and hence γ_\perp maps $H^{\text{div}}(\Omega)$ onto \mathcal{W}' . \square

We can now recover [GR86, Thm I.2.5 and Cor. I.2.8] by taking $\Gamma_0 = \emptyset$:

Corollary A.9. *The normal trace mapping $u \mapsto \nu \cdot \gamma_0 u : \mathcal{D}(\overline{\Omega})^n \rightarrow L^2(\partial\Omega)$ has a unique continuous extension γ_\perp that maps $H^{\text{div}}(\Omega)$ boundedly onto $H^{-1/2}(\partial\Omega)$.*

B Two general operator-technical lemmas

We apply the following lemmas in the proof of Theorem 2.2:

Lemma B.1. *Let T be a closed linear operator from $\text{dom}(T) \subset X$ into Y , where X and Y are Hilbert spaces. Equip $\text{dom}(T)$ with the graph norm of T , in order to make it a Hilbert space. Let R be a restriction of the operator T .*

The closure of the operator R is $\overline{R} = T|_{\overline{\text{dom}(R)}}$, where $\overline{\text{dom}(R)}$ is the closure of $\text{dom}(R)$ in the graph norm of T . In particular, R is a closed operator if and only if $\text{dom}(R)$ is closed in the graph norm of T .

Proof. The following chain of equivalences, where $G(R) = \begin{bmatrix} I \\ R \end{bmatrix} \text{dom}(R)$ denotes the graph of R , proves that $\overline{R} = T|_{\overline{\text{dom}(R)}}$:

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} \in G(\overline{R}) &\stackrel{(i)}{\iff} \exists x_k \in \text{dom}(R) : x_k \xrightarrow{X} x, Rx_k \xrightarrow{Y} y \\ &\stackrel{(ii)}{\iff} \exists x_k \in \text{dom}(R) : x_k \xrightarrow{X} x, Tx_k \xrightarrow{Y} y \\ &\stackrel{(iii)}{\iff} \exists x_k \in \text{dom}(R) : x_k \xrightarrow{\text{dom}(T)} x, Tx = y \\ &\iff x \in \overline{\text{dom}(R)}, Tx = y, \end{aligned}$$

where we have used that (i): $G(\overline{R}) = \overline{G(R)}$ by the definition of operator closure, (ii): $G(R) \subset G(T)$, and (iii): T is continuous from $\text{dom}(T)$ into Y and $\text{dom}(T)$ is complete.

Now it follows easily that R is closed if and only if $\text{dom}(R)$ is closed in $\text{dom}(T)$:

$$R = \overline{R} \implies T|_{\text{dom}(R)} = T|_{\text{dom}(\overline{R})} \implies \text{dom}(R) = \overline{\text{dom}(R)},$$

and moreover, assuming instead that $\text{dom}(R) = \overline{\text{dom}(R)}$, we obtain that

$$R = T|_{\text{dom}(R)} = T|_{\overline{\text{dom}(R)}} = \overline{R}.$$

□

Lemma B.2. *Let γ be a linear operator from the Hilbert space \mathcal{T} into the Hilbert space \mathcal{Z} .*

1. *Let \mathcal{R} and \mathcal{R}' be two linear subspaces of \mathcal{T} such that $\ker(\gamma) \subset \mathcal{R} \cap \mathcal{R}'$ then*

$$\gamma\mathcal{R} = \gamma\mathcal{R}' \quad \text{if and only if} \quad \mathcal{R} = \mathcal{R}'. \quad (\text{B.1})$$

2. *Let \mathcal{R} be a linear subspace of \mathcal{T} and assume that $\gamma : \mathcal{T} \rightarrow \mathcal{Z}$ is continuous and surjective with $\ker(\gamma) \subset \mathcal{R}$. Then $\gamma\overline{\mathcal{R}} = \overline{\gamma\mathcal{R}}$. Furthermore, \mathcal{R} is closed in \mathcal{T} if and only if $\gamma\mathcal{R}$ is closed in \mathcal{Z} .*

Proof. 1. First assume that $\gamma\mathcal{R} = \gamma\mathcal{R}'$ and choose $x \in \mathcal{R}'$ arbitrarily. Then we can find a $\xi \in \mathcal{R}$ such that $\gamma x = \gamma\xi$, and then $x - \xi \in \ker(\gamma) \subset \mathcal{R}$ by the assumption $\ker(\gamma) \subset \mathcal{R} \cap \mathcal{R}'$, so that $x = x - \xi + \xi \in \mathcal{R}$. This proves that $\mathcal{R}' \subset \mathcal{R}$, and since there is no distinction between \mathcal{R} and \mathcal{R}' in the result statement, it also holds that $\mathcal{R} \subset \mathcal{R}'$. The implication from right to left in (B.1) is trivial.

2. Fix $f \in \overline{\mathcal{R}}$ arbitrarily and let $f_k \in \mathcal{R}$ tend to f in \mathcal{T} . Then $\gamma f_k \rightarrow \gamma f$ in \mathcal{Z} by the continuity of γ , and hence $\gamma f \in \overline{\gamma\mathcal{R}}$, i.e., $\gamma\overline{\mathcal{R}} \subset \overline{\gamma\mathcal{R}}$.

For the converse inclusion, we first remark that since γ is onto \mathcal{Z} , there exists a continuous right inverse of γ . We denote this right-inverse by γ^{-r} . Now let $g \in \overline{\gamma\mathcal{R}}$ be arbitrary and let $f_k \in \mathcal{R}$ be a sequence such that $\gamma f_k \rightarrow g$ in \mathcal{Z} . Then $\gamma^{-r}\gamma f_k \rightarrow \gamma^{-r}g =: f'$ in \mathcal{T} and thus $\gamma f' = g$. By the definition of a right inverse there holds $\gamma(f_k - \gamma^{-r}\gamma f_k) = 0$, and since $\ker(\gamma) \subset \mathcal{R}$ and $f_k \in \mathcal{R}$, we conclude that $\gamma^{-r}\gamma f_k \in \mathcal{R}$ for all k . So $f' \in \overline{\mathcal{R}}$ and thus $g = \gamma f' \in \gamma\overline{\mathcal{R}}$. Combining this with the other inclusion, we have established that $\gamma\overline{\mathcal{R}} = \overline{\gamma\mathcal{R}}$. We concentrate next on the last assertion.

If \mathcal{R} is closed, then by the previous result $\gamma\mathcal{R} = \gamma\overline{\mathcal{R}} = \overline{\gamma\mathcal{R}}$. Thus $\gamma\mathcal{R}$ is closed.

If $\gamma\mathcal{R}$ is closed, then $\gamma\mathcal{R} = \overline{\gamma\mathcal{R}} = \gamma\overline{\mathcal{R}}$, where we used the first result again. Defining \mathcal{R}' as $\overline{\mathcal{R}}$, we see that the left hand-side of (B.1) holds. Since $\ker(\gamma) \subset \mathcal{R} = \mathcal{R} \cap \mathcal{R}'$, we conclude from part (1) that $\mathcal{R} = \overline{\mathcal{R}}$. \square